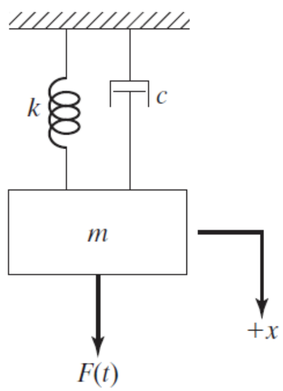
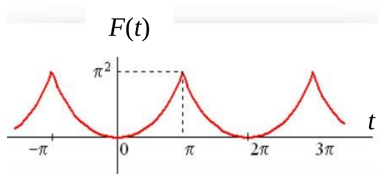
Final exam of Mechanical Vibration		University of Hormozgan
Name:	2020-2021-2	Dr. Mohammad Hosseini
Time: 160 min		Department of Mechanical Engineering

1. Consider a single-degree-of-freedom system subjected to a force $F(t)$, as indicated in figure. Find the total response of the mass.



$F(t)=t^2 \quad -\pi \le t \le \pi$



$m = 5kg, c = 10\dfrac{N.s}{m}$

$k = 100\dfrac{N}{m}$

$x(0) = 0, \dot{x}(0) = 1$

۱- سیستم یک درجه آزادی روبه‌رو را در نظر بگیرید که تحت نیروی $F(t)$ قرار دارد. این نیرو در شکل نشان داده شده است.

پاسخ کلی سیستم را به دست آورید.

توجه: فقط **دو جمله اول** سری را در نظر بگیرید.

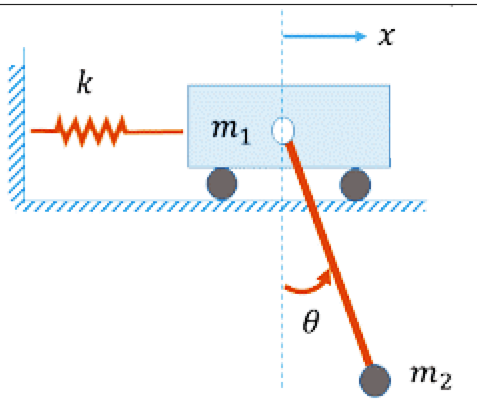
تمام جزئیات محاسبه پاسخ را ذکر کنید.

استفاده از فرمول‌های آماده به هیچ عنوان قابل قبول نیست.

Find the response of the system under $F(t)$.

۷۰ نمره

2. Find the natural frequencies and **mode shapes** of system, shown in figure (Length of massless bar $=L$).



$m_1 = 5kg$

$m_2 = 12kg$

$k = 1000\dfrac{N}{m}$

$L = 0.25m$

۲- فرکانس‌های طبیعی و شکل مودهای سیستم نشان داده شده در شکل را به دست آورید.

میله بدون وزن است و طول آن L است.

تمام جزئیات محاسبه پاسخ را ذکر کنید.

استفاده از فرمول‌های آماده به هیچ عنوان قابل قبول نیست.

حل به صورت پارامتری نیز قابل قبول است.

۳۰ نمره

Obtain the natural frequencies and mode shapes.

زمان آزمون ۱۶۰ دقیقه است. ۲۰ دقیقه زمان اضافه برای آپلود پاسخ‌ها درنظر گرفته شده است. در حین حل سوال دوم، پاسخ سوال اول را آپلود کنید.

منظم و تمیز بنویسید.

فقط پاسخ‌های که در سامانه ارسال شوند، مورد بررسی قرار خواهند گرفت.

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1399_1400_2 پایانترم ارتعاشات

$$F(t) = \frac{a_0}{2} \sum \left[a_n \cos \frac{n\pi}{L} t + b_n \sin \frac{n\pi}{L} t \right]$$

معادله حرکت 10 نمره
ضرایب سری فوریه 20 نمره
حل معادله همگن 15 نمره
حل معادله برای جزء اول سری 15 نمره
- - - - - دوم - 15 نمره
پاسخ کلی 15 نمره

$$a_0 = \frac{1}{L} \int_{-L}^{L} F(t)\, dt$$

$$a_n = \frac{1}{L} \int_{-L}^{L} F(t) \cos \frac{n\pi}{L} t\, dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} F(t) \sin \frac{n\pi}{L} t\, dt$$

$$F(t) = t^2 \qquad -\pi \leq t \leq \pi \qquad L = \pi$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 dt = \frac{1}{\pi} \frac{t^3}{3} \Big|_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$ 5 نمره

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \cos \frac{n\pi}{\pi} t\, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \cos nt\, dt$$

$$\begin{array}{ccc} t^2 & \xrightarrow{+} & \cos nt \\ 2t & \xrightarrow{-} & \frac{1}{n} \sin nt \\ 2 & \xrightarrow{+} & -\frac{1}{n^2} \cos nt \\ 0 & & -\frac{1}{n^3} \sin nt \end{array}$$

$$= \frac{1}{\pi} \left[\frac{1}{n} t^2 \sin nt + \frac{2}{n^2} t \cos nt - \frac{2}{n^3} \sin nt \right]_{-\pi}^{\pi} = \frac{2}{\pi n^2} \left[\pi \cos n\pi - (-\pi) \cos(-n\pi) \right]$$

$$= \frac{2}{\pi n^2} \pi (\cos n\pi + \cos n\pi) = \frac{4}{\pi n^2} \pi \cos n\pi = \frac{4}{n^2} (-1)^n = a_n$$ 5 نمره

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \sin \frac{n\pi}{\pi} t\, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \sin nt\, dt = 0$$ 5 نمره

$$F(t) = t^2 \simeq \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{4}{n^2} (-1)^n \cos nt \right] = \frac{\pi^2}{3} + \left(-4\cos t + \cos 2t - \frac{4}{9} \cos 3t + \cdots \right.$$

5 نمره

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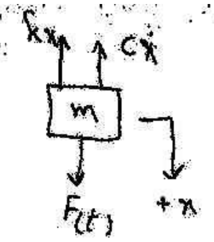

$$-kx - c\dot{x} + F(t) = m\ddot{x} \Rightarrow m\ddot{x} + c\dot{x} + kx = F(t)$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = \frac{\pi^2}{3} - 4\cos t \Rightarrow \quad \boxed{10}$$

$$\begin{cases} m\ddot{x} + c\dot{x} + kx = 0 \\ m\ddot{x} + c\dot{x} + kx = \frac{\pi^2}{3} \\ m\ddot{x} + c\dot{x} + kx = -4\cos t \end{cases} \qquad m = 5\ kg, \quad c = 10\ \frac{N.s}{m}, \quad k = 100\ N/m$$

$\boxed{1}$ $m\ddot{x} + c\dot{x} + kx = 0 \Rightarrow x_h(t) = \overset{\text{constant}}{A} e^{st} \rightarrow \dot{x}_h(t) = Ase^{st} \rightarrow \ddot{x}_h(t) = As^2e^{st}$

$$m(As^2e^{st}) + c(Ase^{st}) + k(Ae^{st}) = 0 \Rightarrow ms^2 + cs + k = 0 \Rightarrow$$

$$s = \frac{-c \pm \sqrt{c^2 - 4 \times m \times k}}{2m} \Rightarrow$$

$$s = -\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m} \Rightarrow c_c = 2\sqrt{km} = 2m\omega_n = 2\sqrt{100 \times 5} = 44.721$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{5}} = 4.472, \quad \zeta = \frac{c}{c_c} = \frac{10}{44.721} = 0.2236 < 1 \rightarrow \text{under damped}$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = \sqrt{\frac{k}{m}(1 - \zeta^2)} = 4.36$$

$$\rightarrow s = \frac{-c \pm m\sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}}}{2m}, \quad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\sqrt{\frac{k}{m}}} = \frac{c}{2m\omega_n} \Rightarrow$$

$$\begin{cases} \frac{c}{m} = 2\zeta\omega_n \\ \frac{c}{2m} = \zeta\omega \end{cases} \Rightarrow s = -\frac{c}{2m} \pm \frac{1}{2}\sqrt{4\left(\frac{c}{2m}\right)^2 - 4\frac{k}{m}} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - \frac{k}{m}}$$

$$= -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = \left(-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}\right) = -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2}$$

$$\boxed{S_{1,2} = \omega_n\left(-\zeta \pm i\sqrt{1 - \zeta^2}\right)}$$

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$$x_h(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{(-\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2})t} + A_2 e^{(-\zeta\omega_n - i\omega_n \sqrt{1-\zeta^2})t}$$

$$\xrightarrow{\omega_d = \omega_n\sqrt{1-\zeta^2}} x_h(t) = A_1 e^{(-\zeta\omega_n + i\omega_d)t} + A_2 e^{(-\zeta\omega_n - i\omega_d)t} =$$

$$e^{-\zeta\omega_n t}\left[A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right]$$

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta \Rightarrow$$

$$x_h(t) = e^{-\zeta\omega_n t}\left[A_1(\cos\omega_d t + i\sin\omega_d t) + A_2(\cos\omega_d t - i\sin\omega_d t)\right]$$

$$= e^{-\zeta\omega_n t}\left[\underbrace{(A_1+A_2)}_{A'_1}\cos\omega_d t + \underbrace{i(A_1-A_2)}_{A'_2}\sin\omega_d t\right] \Rightarrow$$

$$x_h(t) = e^{-\zeta\omega_n t}\left[A'_1\cos\omega_d t + A'_2\sin\omega_d t\right]$$ $\boxed{\text{10 نمره}}$

$\boxed{2}$ $m\ddot{x} + c\dot{x} + kx = \frac{\pi^2}{3}$ $\Rightarrow x_{1p}(t) = A_3 = \text{constant} \Rightarrow$

$\dot{x}_{1p}(t) = 0 \Rightarrow \ddot{x}_{1p}(t) = 0 \Rightarrow kA_3 = \frac{\pi^2}{3} \Rightarrow A_3 = \frac{\pi^2}{3k}$

$\Rightarrow \boxed{x_{1p}(t) = \frac{\pi^2}{3k}}$ $\boxed{\text{10 نمره}}$

$\boxed{3}$ $m\ddot{x} + c\dot{x} + kx = -4\cos x = \alpha_1 \cos\omega_1 t$ $\begin{cases} \alpha_1 = -4 \\ \omega_1 = 1 \end{cases}$

$x_{2p}(t) = A_4 \sin\omega_1 t + A_5 \cos\omega_1 t \Rightarrow$

$\dot{x}_{2p}(t) = A_4\omega_1 \cos\omega_1 t - A_5\omega_1 \sin\omega_1 t \Rightarrow$

$\ddot{x}_{2p}(t) = -A_4\omega_1^2 \sin\omega_1 t - A_5\omega_1^2 \cos\omega_1 t \Rightarrow$

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$$m\left[-A_4\omega_1^2\sin\omega_1 t - A_5\omega_1^2\cos\omega_1 t\right] + c\left[A_4\omega_1\cos\omega_1 t - A_5\omega_1\sin\omega_1 t\right]$$

$$+ k\left[A_4\sin\omega_1 t + A_5\cos\omega_1 t\right] = a_1\cos\omega_1 t \Rightarrow$$

$$\left[-m\omega_1^2 A_4 - c\omega_1 A_5 + kA_4\right]\sin\omega_1 t + \left[-m\omega_1^2 A_5 + c\omega_1 A_4 + kA_5\right]\cos\omega_1 t = a_1\cos\omega_1 t$$

$$\Rightarrow \left[(k-m\omega_1^2)A_4 - c\omega_1 A_5\right]\sin\omega_1 t + \left[(k-m\omega_1^2)A_5 + c\omega_1 A_4\right]\cos\omega_1 t = a_1\cos\omega_1 t + 0\sin\omega_1 t$$

$$\begin{array}{l} \sin\omega_1 t \\ \cos\omega_1 t \end{array}\begin{cases} (k-m\omega_1)^2 A_4 - c\omega_1 A_5 = 0 \\ c\omega_1 A_4 + (k-m\omega_1^2)A_5 = a_1 \end{cases} \quad \text{or} \begin{bmatrix} (k-m\omega_1^2) & -c\omega_1 \\ c\omega_1 & (k-m\omega_1^2) \end{bmatrix}\begin{Bmatrix} A_4 \\ A_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ a_1 \end{Bmatrix} \Rightarrow$$

$$\sin\omega_1 t \begin{Bmatrix} A_4 \\ A_5 \end{Bmatrix} = \frac{1}{(k-m\omega_1^2)^2 + (c\omega_1)^2}\begin{bmatrix} (k-m\omega_1^2) & c\omega_1 \\ -c\omega_1 & (k-m\omega_1^2) \end{bmatrix}\begin{Bmatrix} 0 \\ a_1 \end{Bmatrix} \Rightarrow$$

$$\& \quad A_5 = \frac{a_1\left[(k-m\omega_1^2)\right]}{(k-m\omega_1^2)^2 + (c\omega_1)^2}, \quad A_4 = \frac{a_1\left[c\omega_1\right]}{(k-m\omega_1^2)^2 + (c\omega_1)^2}$$

$$\Rightarrow x_{2p}(t) = \frac{(k-m\omega_1^2)\,a_1}{(k-m\omega_1^2)^2 + (c\omega_1)^2}\cos\omega_1 t + \frac{(c\omega_1)\,a_1}{(k-m\omega_1^2)^2 + (c\omega_1)^2}\sin\omega_1 t$$

$\boxed{\text{نمره 10}}$

$$x(t) = x_h(t) + x_{1p}(t) + x_{2p}(t)$$

$$= e^{-\zeta\omega_n t}\left[A_1'\cos\omega_d t + A_2'\sin\omega_d t\right] + \frac{\pi^2}{3k}$$

$$+ \frac{(k-m\omega_1^2)}{(k-m\omega_1^2)^2 + (c\omega_1)^2}a_1\cos\omega_1 t + \frac{(c\omega_1)\,a_1}{(k-m\omega_1^2)^2 + (c\omega_1)^2}\sin\omega_1 t$$

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$x_{(0)} = x_0 \Rightarrow$

$$A_1' + \frac{\pi^2}{3k} + \frac{(k - m\omega_1^2)\,a_1}{(k-m\omega_1^2)^2 + (c\omega_1)^2} = x_0 \Rightarrow$$

$$\boxed{A_1' = x_0 - \frac{(k - m\omega_1^2)\,a_1}{(k-m\omega_1^2)^2 + (c\omega_1)^2} - \frac{\pi^2}{3k}}$$

$$\dot{x}_{(t)} = -\zeta\omega_n e^{-\zeta\omega_n t}\left[A_1'\cos\omega_d t + A_2'\sin\omega_d t\right] + \omega_d e^{-\zeta\omega_n t}\left[-A_1'\sin\omega_d t + A_2'\cos\omega_d t\right]$$

$$- \frac{(k - m\omega_1^2)}{(k-m\omega_1^2)^2 + (c\omega_1)^2}\, a_1\omega_1 \sin\omega_1 t + \frac{c\omega_1}{(k-m\omega_1^2)^2 + (c\omega_1)^2}\, a_1\omega_1 \cos\omega_1 t$$

$$\dot{x}_{(0)} = \dot{x}_0 \Rightarrow -\zeta\omega_n A_1' + \omega_d A_2' + \frac{c\omega_1}{(k-m\omega_1^2)^2 + (c\omega_1)^2}\, a_1\omega_1 = \dot{x}_0 \Rightarrow$$

$$A_2' = \frac{1}{\omega_d}\left[\dot{x}_0 + \zeta\omega_n\left[x_0 - \frac{(k - m\omega_1^2)\,a_1}{(k-m\omega_1^2)^2 + (c\omega_1)^2} - \frac{\pi^2}{3k}\right] - \frac{c\omega_1}{(k-m\omega_1^2)^2 + (c\omega_1)^2}\, a_1\omega_1\right]$$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{5}} = 4.4721$, $C_c = 2m\omega_n = 2\sqrt{km} = 2\sqrt{100 \times 5} = 44.7214$

$\zeta = \frac{c}{c_c} = 0.2236$, $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4.3589$

$A_4' = -0.0044$, $A_5 = -0.0416$, $A_1' = 0.0087$, $A_2' = 0.2324$

$$x_{(t)} = e^{-t}\left[0.0087\cos 4.3589t + 0.2324\sin 4.3589t\right] + 0.0329$$

$$-0.0044\sin\omega t - 0.0416\cos\omega t$$ پاسخ

or: $x_{(t)} = 0.232563\, e^{-t}\sin(4.3589t + 0.0374) + 0.0329 - 0.04183\sin(t + 1.4634)$

استفاده از تبدیل لاپلاس

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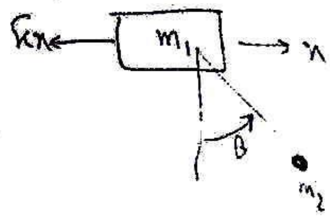
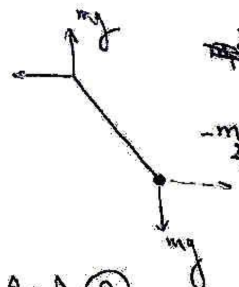
$$-kx = m_1\ddot{x} + m_2(\ddot{x} + L\ddot{\theta})$$





$$(m_1+m_2)\ddot{x} + m_2L\ddot{\theta} + kx = 0 \quad (1)$$

$$-m_2gL\theta = J\left(\ddot{\theta} + \frac{\ddot{x}}{L}\right), \quad J = m_2L^2$$

$$m_2L\ddot{x} + m_2L^2\ddot{\theta} + m_2gL\theta = 0 \quad (2)$$

$$\begin{bmatrix} (m_1+m_2) & m_2L \\ m_2L & m_2L^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & m_2gL \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \boxed{0.5} \Rightarrow \begin{cases} x(t) = X\cos(\omega t + \phi) \\ \theta(t) = \Theta\cos(\omega t + \phi) \end{cases}$$

$$\begin{bmatrix} k-(m_1+m_2)\omega^2 & -m_2L\omega^2 \\ -m_2L\omega^2 & m_2gL - m_2L^2\omega^2 \end{bmatrix}\begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (I) \quad \boxed{0.5}$$

$$[k-(m_1+m_2)\omega][m_2L(g-L\omega^2)] - m_2^2L^2\omega^4 = 0 \Rightarrow$$

$$km_2Lg \underbrace{- km_2L^2\omega^2 - m_2gL(m_1+m_2)\omega^2} + \underbrace{m_2L^2(m_1+m_2)\omega^4 - m_2^2L^2\omega^4} = 0$$

$$[m_2L^2(m_1+m_2) - m_2^2L^2]\omega^4 - m_2L[kL + g(m_1+m_2)]\omega^2 + km_2Lg = 0$$

$$[k-(m_1+m_2)\omega_1^2]X_1^{(1)} - m_2L\omega_1^2\Theta^{(1)} = 0 \Rightarrow r_1^{(1)} = \frac{\Theta^{(1)}}{X^{(1)}} = \frac{k-(m_1+m_2)\omega_1^2}{m_2L\omega_1^2}$$

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$$L\left[m_1+m_2-m_2\right]\omega^4-\left[kL+g(m_1+m_2)\right]\omega^2+kg=0$$

$$\omega^2=\frac{\left[kL+g(m_1+m_2)\right]\pm\sqrt{\left[kL+g(m_1+m_2)\right]^2-4m_1Lkg}}{2\times m_1L}$$

$\omega_1^2=307.9297$, $\omega_2^2=25.4863$

$\omega_1=17.5479258$ $\qquad$ $\omega_2=5.048395785$ $\qquad$ (II)

(II) into (I)

$$I:\begin{bmatrix}1000-17\omega^2 & -3\omega^2\\ -3\omega^2 & 29.43-0.75\omega^2\end{bmatrix}\begin{Bmatrix}X\\ \theta\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

$$\omega=\omega_1\Rightarrow\begin{bmatrix}1000-17\times307.9297 & -3\times307.9297\\ -3\times307.9297 & 29.43-0.75\times307.9297\end{bmatrix}\begin{Bmatrix}X^{(1)}\\ \theta^{(1)}\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

$$-4234.8049\,X^{(1)}-923.7891\,\theta^{(1)}=0\Rightarrow r_1=\frac{\theta^{(1)}}{X^{(1)}}=-\frac{4231.8049}{923.7891}=-4.5809$$

$$\text{or } r_1=-0.21829=\frac{X^{(1)}}{\theta^{(1)}}$$

$$\omega=\omega_2=\begin{bmatrix}1000-17\times25.4863 & -3\times25.4863\\ -3\times25.4863 & 29.43-0.75\times25.4863\end{bmatrix}\begin{Bmatrix}X^{(2)}\\ \theta^{(2)}\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

$$566.7329\,X^{(2)}-76.4589\,\theta^{(2)}=0\Rightarrow r_2=\frac{\theta^{(2)}}{X^{(2)}}=\frac{566.7329}{76.4589}=7.4152$$

$$\text{or } r_2=\frac{X^{(2)}}{\theta^{(2)}}=\frac{76.4589}{566.7329}=0.134911$$

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$$\vec{X}^{(1)} = \begin{Bmatrix} X^{(1)} \\ \Theta^{(1)} \end{Bmatrix} = \begin{Bmatrix} X^{(1)} \\ r_1 X^{(1)} \end{Bmatrix}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} X^{(2)} \\ \Theta^{(2)} \end{Bmatrix} = \begin{Bmatrix} X^{(2)} \\ r_2 X^{(2)} \end{Bmatrix}$$

Normal modes

$$\underbrace{\vec{x}^{(1)}(t) : \begin{Bmatrix} X^{(1)} \cos(\omega t + \phi) \\ r_1 X^{(1)} \cos(\omega t + \phi) \end{Bmatrix}}_{\text{First mode}} , \quad \underbrace{\vec{x}^{(2)}(t) = \begin{Bmatrix} X^{(2)} \cos(\omega t + \phi) \\ r_2 X^{(2)} \cos(\omega t + \phi) \end{Bmatrix}}_{\text{Second Mode}}$$

$\boxed{5\ \text{نمره}}$

Where the constants $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 and ϕ_2 are determined by the initial condibions.

روش لاگرانژ درس داده نشده است! چگونه دانشجویان از این روش استفاده کرده اند!!

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The kinetic energy: $T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left[\frac{d}{dt}(x + l\sin\theta)\right]^2 \Rightarrow$

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left[\dot{x} + l\dot{\theta}\cos\theta\right]^2 = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left[\dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\cos^2\theta\right]$$

$$T = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2} m_2 \left[2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\cos^2\theta\right]$$

- The potential energ: $U = \frac{1}{2} k x^2 - m_2 g l \cos\theta$

$$\frac{\partial T}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2 l\dot{\theta}\underbrace{\cos\theta}_{1} \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = (m_1+m_2)\ddot{x} + m_2 l\ddot{\theta}\underbrace{\cos\theta}_{1} - \underbrace{m_2 l\dot{\theta}^2\underbrace{\sin\theta}_{\theta}}_{0}$$

$$\frac{\partial T}{\partial \dot{\theta}} = m_2 l\dot{x}\underbrace{\cos\theta}_{1} + m_2 l^2\dot{\theta}\underbrace{\cos\theta}_{1} \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = m_2 l\ddot{x}\underbrace{\cos\theta}_{1} - \underbrace{m_2 l\dot{x}\dot{\theta}\underbrace{\sin\theta}_{\theta}}_{0} + m_2 l^2\ddot{\theta}\underbrace{\cos\theta}_{1} - \underbrace{m_2 l^2\dot{\theta}^2\underbrace{\sin\theta}_{1}}_{0}$$

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial \theta} = \frac{1}{2} m_2 \left[-2l\dot{x}\dot{\theta}\underbrace{\sin\theta}_{\theta} - 2l^2\dot{\theta}^2\underbrace{\sin\theta}_{\theta}\underbrace{\cos\theta}_{1}\right] \xrightarrow{\theta \text{ is very smal } (\theta \ll 1)} 0$$

$$\frac{\partial U}{\partial x} = kx, \quad \frac{\partial U}{\partial \theta} = m_2 g l \sin\theta \simeq m_2 g l \theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0, \quad i = 1, 2 \Rightarrow$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = 0 \Rightarrow (m_1+m_2)\ddot{x} + m_2 l\ddot{\theta} - 0 + kx = 0 \Rightarrow \quad \text{(I)}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = 0 \Rightarrow m_2 l\ddot{x} + m_2 l^2\ddot{\theta} - 0 + m_2 g l\theta = 0 \quad \text{(II)}$$

$$\text{(I) and (II)} \Rightarrow \begin{bmatrix} m_1+m_2 & m_2 l \\ m_2 l & m_2 l^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & m_2 g l \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

جالب است که این روش در فصل ۶ است و درس داده نشده ولی همه دانشجوها از این روش استفاده کرده اند!!!