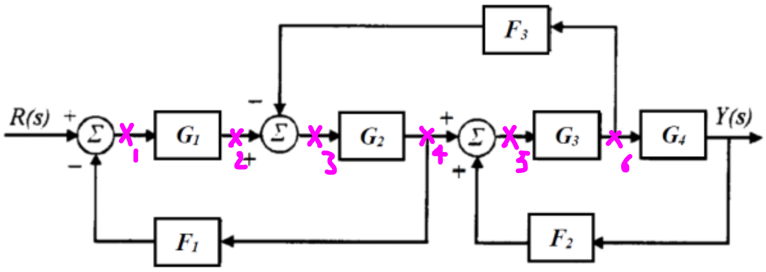
Find the transfer function of block diagram.



$X_1 = R - F_1 X_4$ $\quad X_2 = G_1 R - F_1 G_1 X_4$ (1)

$X_2 = G_1 X_1$

$X_3 = X_2 - F_3 X_6$ $\quad X_3 = (G_1 R - F_1 G_1 X_4) - F_3 X_6$ (2)

$X_4 = G_2 X_3$

$X_4 = G_1 G_2 R - F_1 G_1 G_2 X_4 - F_3 G_2 X_6$ (3)

$X_5 = X_4 + F_2 Y$

$Y = G_3 G_4 X_4 + G_3 G_4 F_2 Y$ (5)

$X_6 = G_3 X_5$

$Y = G_4 X_6$ $\quad Y = G_3 G_4 X_5$ (4)

(5) $\Rightarrow (1 - F_2 G_3 G_4) Y = G_3 G_4 X_4 \Rightarrow X_4 = \dfrac{1 - F_2 G_3 G_4}{G_3 G_4} Y$ (6)

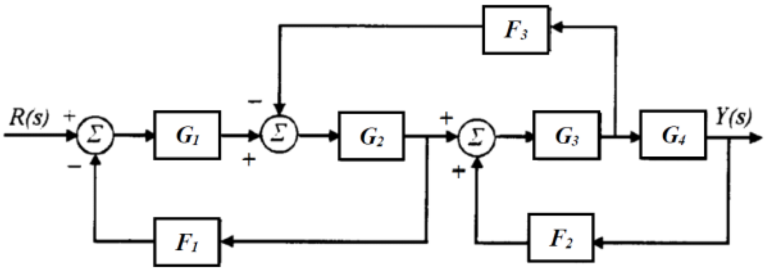
(6) into (3) $\Rightarrow$ $\dfrac{1 - F_2 G_3 G_4}{G_3 G_4} Y = G_1 G_2 R - F_1 G_1 G_2 \dfrac{1 - F_2 G_3 G_4}{G_3 G_4} Y - F_3 G_2 \dfrac{Y}{G_4} \Rightarrow$

$$\left[\frac{1 - F_2 G_3 G_4}{G_3 G_4} + \frac{F_1 G_1 G_2 - F_1 F_2 G_1 G_2 G_3 G_4}{G_3 G_4} + \frac{F_3 G_2}{G_4}\right] Y = G_1 G_2 R \Rightarrow$$

$$\left[\frac{1 - F_2 G_3 G_4 + F_1 G_1 G_2 - F_1 F_2 G_1 G_2 G_3 G_4 + F_3 G_2 G_3}{G_3 G_4}\right] Y = G_1 G_2 R \Rightarrow$$

$$TF = \frac{Y}{R} = \frac{G_1 G_2 G_3 G_4}{1 - F_2 G_3 G_4 + F_1 G_1 G_2 - F_1 F_2 G_1 G_2 G_3 G_4 + F_3 G_2 G_3}$$

تابع تبدیل



مسیر پیشرو $G_1G_2G_3G_4$ $\quad\Delta = 1$

حلقه‌های تکی: $-G_1G_2F_1$ و $G_3G_4F_2$ و $-G_2G_3F_3$

حلقه‌های دوتایی: $(-G_1G_2F_1)(G_3G_4F_2)$

$$TF = \frac{G_1G_2G_3G_4}{1-[-F_1G_1G_2+F_2G_3G_4-F_3G_2G_3]+(-F_1F_2G_1G_2G_3G_4)} \Rightarrow$$

$$TF = \frac{G_1G_2G_3G_4}{1+F_1G_1G_2-F_2G_3G_4+F_3G_2G_3-F_1F_2G_1G_2G_3G_4}$$

2. Consider the following state space equations.
Determine the response of the system for $x(0)=\begin{pmatrix}1\\-1\end{pmatrix}$

۲- سیستم زیر را در نظر بگیرید (۳۰ نمره).

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

$$y = [1 \quad 0]\mathbf{x}$$

پاسخ سیستم را به ازای $x(0)=\begin{pmatrix}1\\-1\end{pmatrix}$ و $u(t)=0$ به دست آورید.

$[A] = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ و $[B] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ و $[C] = [1 \quad 0]$ و $[D] = 0$

$\Phi(s) = (sI - A)^{-1}$

$$\Phi_{(t)} = e^{At} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}$$

$$x_{(t)} = \Phi_{(t)}\, x_{(0)} + \int_0^t \Phi_{(t-\tau)} B\, u_{(\tau)}\, d\tau$$

$$X_{(s)} = (sI-A)^{-1} x_{(0)} + (sI-A)^{-1} B\, U_{(s)} = \Phi_{(s)}\, x_{(0)} + \Phi_{(s)} B\, U_{(s)}$$

$$sI-A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{\underbrace{s^2+3s+2}_{(s+1)(s+2)}} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ -\frac{2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\frac{s+3}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{B_1}{s+2} \Rightarrow s+3 = A_1 s + 2A_1 + B_1 s + B_1 = (A_1+B_1)s + (2A_1+B_1)$$

$$\begin{cases} A_1+B_1 = 1 \\ 2A_1+B_1 = 3 \end{cases} \Rightarrow A_1 = 2 \text{ and } B_1 = -1$$

$$\boxed{\frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A_2}{s+1} + \frac{B_2}{s+2} \Rightarrow 1 = (A_2+B_2)s + (2A_2+B_2) \Rightarrow \begin{cases} A_2+B_2 = 0 \\ 2A_2+B_2 = 1 \end{cases} \Rightarrow \begin{matrix} A_2 = 1 \\ B_2 = -1 \end{matrix}$$

$$\boxed{\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}}$$

$$\boxed{-\frac{2}{(s+1)(s+2)} = -\frac{2}{s+1} + \frac{2}{s+2}}$$

$$\frac{s}{(s+1)(s+2)} = \frac{A_4}{s+1} + \frac{B_4}{s+2} \Rightarrow s = (A_4+B_4)s + (2A_4+B_4) \Rightarrow \begin{cases} A_4+B_4 = 1 \\ 2A_4+B_4 = 0 \end{cases} \Rightarrow \begin{matrix} A_4 = -1 \\ B_4 = 2 \end{matrix}$$

$$\boxed{\frac{s}{(s+1)(s+2)} = \frac{-1}{s+1} + \frac{2}{(s+2)}}$$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ -\frac{2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$\varphi_{(t)} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$x_{(t)} = \Phi_{(t)} x_{(0)} + \int_0^t \varphi_{(t-\tau)} B u_{(\tau)} \, d\tau$$

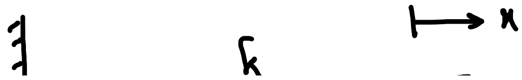
$$x_{(t)} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 2e^{-t} - e^{-2t} - e^{-t} + e^{-2t} \\ -2e^{-t} + 2e^{-2t} + e^{-t} - 2e^{-2t} \end{Bmatrix}$$

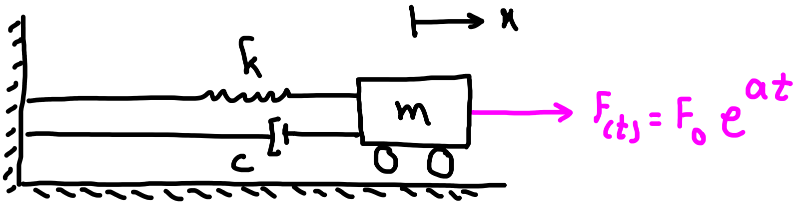
$$\boxed{x_{(t)} = \begin{Bmatrix} e^{-t} \\ -e^{-t} \end{Bmatrix}}$$

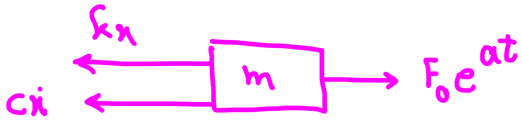
پاسخ سیستم

$$y = [1 \quad 0]\, x \Rightarrow y = [1 \quad 0] \begin{Bmatrix} e^{-t} \\ -e^{-t} \end{Bmatrix} = e^{-t} \Rightarrow \boxed{y_{(t)} = e^{-t}}$$

3. Assuming that the system is under damped and $F(t)=F_0e^{at}$, find the state response steady state response.







$$\Sigma F_x = m\ddot{x} \Rightarrow -kx - c\dot{x} + F_0 e^{at} = m\ddot{x} \Rightarrow \boxed{m\ddot{x} + c\dot{x} + kx = F_0 e^{at}}$$

معادله حرکت

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}e^{at} \Rightarrow \boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}e^{at}}$$

اکنون باید تبدیل لاپلاس برای محاسبه پاسخ استفاده می‌کنیم

$$\mathcal{L}\{\ddot{x}\} + 2\zeta\omega_n\mathcal{L}\{\dot{x}\} + \omega_n^2\mathcal{L}\{x\} = \frac{F_0}{m}\mathcal{L}\{e^{at}\}$$

$$\mathcal{L}\{x\} = X_{(s)} \;,\; \mathcal{L}\{\dot{x}\} = sX_{(s)} - x_0 \;,\; \mathcal{L}\{\ddot{x}\} = s^2X_{(s)} - sx_{(0)} - \dot{x}_{(0)}$$

$$\to s^2X - sx_{(0)} - \dot{x}_{(0)} + 2\zeta\omega_n(sX - x_0) + \omega_n^2X = \frac{F_0}{m}\times\frac{1}{s-a}$$

$x_{(0)} = 0 \;,\; \dot{x}_{(0)} = 0$ با فرض شرایط اولیه صفر داریم

$$X(s^2 + 2\zeta\omega_n s + \omega_n^2) = \frac{F_0}{m}\frac{1}{s-a} \Rightarrow X = \frac{F_0}{m}\times\frac{1}{(s^2+2\zeta\omega_n s+\omega_n^2)(s-a)}$$

$$\frac{1}{(s^2+2\zeta\omega_n s+\omega_n^2)(s-a)} = \frac{As+B}{s^2+2\zeta\omega_n s+\omega_n^2} + \frac{C}{s-a} \Rightarrow$$

$$1 = As^2 - aAs + Bs - Ba + Cs^2 + 2\zeta\omega_n Cs + C\omega_n^2 \Rightarrow$$

$$1 = (A+C)s^2 + (-aA + B + 2\zeta\omega_n C)s + (-Ba + C\omega_n^2) \Rightarrow \begin{cases} -Ba + C\omega_n^2 = 1 & (1) \\ -aA + B + 2\zeta\omega_n C = 0 & (2) \end{cases}$$

$$1 = (A+C)s^2 + (-aA + B + 2\zeta\omega_n C)s + (-Ba + C\omega_n^2) \Rightarrow \begin{cases} -aA + B + 2\zeta\omega_n C = 0 & (2) \\ A + C = 0 & (3) \end{cases}$$

$$\left.\begin{array}{l} (3) \Rightarrow A = -C \\ (1) \Rightarrow B = \dfrac{\omega_n^2}{a}C - \dfrac{1}{a} \end{array}\right\} \text{into } (2) \Rightarrow -a(-C) + \left[\frac{\omega_n^2}{a}C - \frac{1}{a}\right] + 2\zeta\omega_n C = 0 \Rightarrow$$

$$\left[a + \frac{\omega_n^2}{a} + 2\zeta\omega_n\right]C = \frac{1}{a} \Rightarrow C = \frac{1}{a\left[a + \frac{\omega_n^2}{a} + 2\zeta\omega_n\right]} = \frac{1}{a^2 + \omega_n^2 + 2\zeta\omega_n a}$$

$$\boxed{A = -\frac{1}{a^2 + \omega_n^2 + 2\zeta\omega_n a}}$$

$$B = \frac{\omega_n^2}{a} \times \frac{1}{a^2 + \omega_n^2 + 2\zeta\omega_n a} - \frac{1}{a} = \frac{\omega_n^2 - [a^2 + \omega_n^2 + 2\zeta\omega_n a]}{a[a^2 + \omega_n^2 + 2\zeta\omega_n a]} = -\frac{a + 2\zeta\omega_n}{a^2 + \omega_n^2 + 2\zeta\omega_n a}$$

$$X = \frac{F_0}{m}\left\{\frac{1}{a^2 + \omega_n^2 + 2\zeta\omega_n a}\left[-\frac{s + (a + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{1}{s-a}\right]\right\} \Rightarrow$$

$$X = \frac{F_0}{m[a^2 + \omega_n^2 + 2\zeta\omega_n a]}\left[\frac{1}{s-a} - \frac{(s + \zeta\omega_n) + (a + \zeta\omega_n)\frac{\omega_d}{\omega_d}}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] \Rightarrow$$

$$X = \frac{F_0}{m[a^2 + \omega_n^2 + 2\zeta\omega_n a]}\left[\frac{1}{s-a} - \left\{\frac{(s + \zeta\omega_n)}{(s + \zeta\omega_n)^2 + \omega_d^2} + \frac{(a + \zeta\omega_n)}{\omega_d}\,\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right\}\right]$$

$$x(t) = \mathcal{L}^{-1}\{X\} = \frac{F_0}{m[a^2 + \omega_n^2 + 2\zeta\omega_n a]}\left[e^{at} - \left\{e^{-\zeta\omega_n t}\cos\omega_d t + \frac{a + \zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\sin\omega_d t\right\}\right]$$

$$\text{or} \quad x(t) = \frac{F_0}{m[a^2 + \omega_n^2 + 2\zeta\omega_n a]}\left[e^{at} - e^{-\zeta\omega_n t}\left\{\cos\omega_d t + \frac{a + \zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right\}\right]$$

$$\cdots \quad m[a^2+\omega_n^2+2\zeta\omega_n a] \quad \cdots \quad \omega_n\sqrt{1-\zeta^2} \quad \cdots$$

قضیه مقدار نهایی

با افزایش زمان دامنه نوسانات به سمت بینهایت میل می‌کند

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\,X(s) = \infty$$