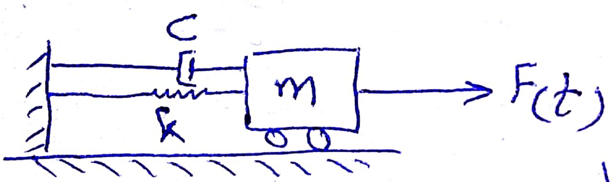
ذکر تمامی جزئیات زیر الزامی است و نمره دارند.

۱- پاسخ کلی سیستم زیر را به ورودی $F(t)$ به دست آورید.



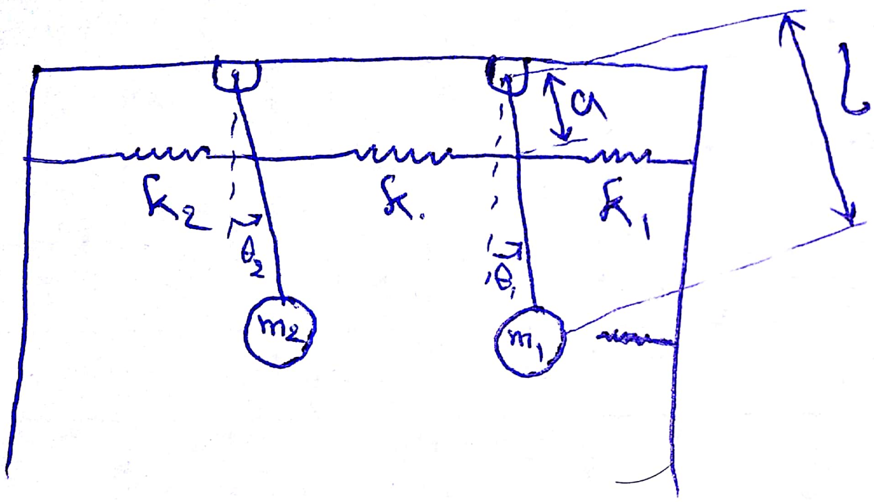
$$F(t) = \begin{cases} 1 & -\pi \leq t < 0 \\ 0 & 0 < t \leq \pi \end{cases}$$

دوره تناوب ← $\boxed{\tau = 2\pi}$

پاسخ کلی را برای دو جمله اول محاسبه کنید.

$m = 5\ kg$ ، $k = 1000\ \frac{N}{m}$ ، $c = 10\ \frac{N.s}{m}$

۲- فرکانس های طبیعی و شکل مودهای سیستم زیر را به دست آورید



$m_1 = 1\ kg$

$m_2 = 2\ kg$

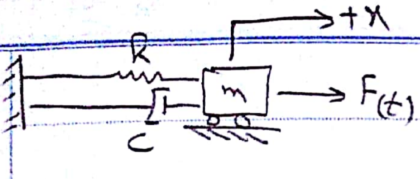
$k_1 = 1000\ N/m$

$k_2 = 2000\ N/m$

$k = 1500\ N/m$

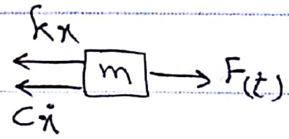
$a = 0.1\ m$

$L = 0.5\ m$



$$F(t) = \begin{cases} 1 & -\pi \le t < 0 \\ 0 & 0 \le t \le \pi \end{cases} \qquad \tau = 2\pi$$

F.B.D.



$$-kx - c\dot{x} + F(t) = m\ddot{x} \Rightarrow$$

$$\boxed{m\ddot{x} + c\dot{x} + kx = F(t)}$$

پاسخ حل عمومی

$$m\ddot{x}+c\dot{x}+kx=0 \qquad x = De^{st} \Rightarrow \dot{x} = Dse^{st} \Rightarrow \ddot{x} = Ds^2e^{st} \Rightarrow$$

$$ms^2De^{st} + csDe^{st} + kDe^{st} = 0 \Rightarrow ms^2+cs+k=0 \Rightarrow$$

$$s = \frac{-c \pm \sqrt{c^2-4mk}}{2m} = \frac{-10 \pm \sqrt{10^2 - 4\times5\times1000}}{2\times5} \Rightarrow$$

$$s = \frac{-10 \pm \sqrt{-19900}}{10} = -1 \pm i\,14.10674 \Rightarrow \begin{cases} s_1 = -1 + 14.10674i \\ s_2 = -1 - 14.10674i \end{cases} \Rightarrow$$

$$x = D_1e^{s_1t} + D_2e^{s_2t} \Rightarrow x = D_1e^{(-1+14.10674i)t} + D_2e^{(-1-14.10674i)t} \Rightarrow$$

$$x = e^{-t}\left[D_1e^{14.10674it} + D_2e^{-14.10674it}\right], \quad e^{\pm i\theta} = \cos\theta \pm i\sin\theta \Rightarrow$$

$$x = e^{-t}\left[D_1\left(\cos(14.10674t) + i\sin(14.10674t)\right) + D_2\left(\cos(14.10674t) - i\sin(14.10674t)\right)\right] \Rightarrow$$

$$x = e^{-t}\left[(D_1+D_2)\cos(14.10674t) + i(D_1-D_2)\sin(14.10674t)\right] \Rightarrow$$

$$\boxed{x_h = x = e^{-t}\left[A_1\cos(14.10674t) + A_2\sin(14.10674t)\right]}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{5}} = 10\sqrt{2} = 14.14214, \quad c_c = 2\sqrt{mk} = 2\sqrt{5\times1000} = 141.4214$$

$$\zeta = \frac{c}{c_c} = \frac{10}{141.4214} = 0.07071 < 1 \Rightarrow \text{underdamped}$$

$$\zeta\omega = 1 \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{1000}{5}}\sqrt{1-0.07071} = \sqrt{199} = 14.10674$$

حل خصوصی را ابتدا سری فوریه مربوط به تابع $F(t)$ را بدست می‌آوریم:

$$F(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi}{\tau}t+b_n\sin\frac{2n\pi}{\tau}t\right)$$

$$a_0=\frac{2}{\tau}\int F(t)\,dt=\frac{2}{2\pi}\int_{-\pi}^{\pi}F(t)\,dt=\frac{2}{2\pi}\left[\int_{-\pi}^{0}(1)\,dt+\int_{0}^{\pi}0\,dt\right]$$

$$=\frac{2}{2\pi}\,t\Big|_{-\pi}^{0}=1\Rightarrow\boxed{a_0=1}$$

$$a_n=\frac{2}{2\pi}\int F(t)\cos\frac{2n\pi}{\tau}t\,dt=\frac{2}{2\pi}\left[\int_{-\pi}^{0}(1)\cos\frac{2n\pi}{2\pi}t\,dt+\int_{0}^{\pi}(0)\cos\frac{2n\pi}{2\pi}t\,dt\right]$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}\cos nt\,dt=\frac{1}{\pi}\left(\frac{1}{n}\sin nt\right)\Big|_{-\pi}^{0}=\frac{1}{\pi}\left(\frac{1}{n}\left(0-\sin(-n\pi)\right)\right)=0\Rightarrow\boxed{a_n=0}$$

$$b_n=\frac{2}{\tau}\int F(t)\sin\frac{2n\pi}{\tau}t\,dt=\frac{2}{2\pi}\left[\int_{-\pi}^{0}(1)\sin\left(\frac{2n\pi}{2\pi}t\right)dt+\int_{0}^{\pi}(0)\sin\left(\frac{2n\pi}{2\pi}t\right)dt\right]$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}\sin(nt)\,dt=\frac{1}{\pi}\left(-\frac{1}{n}\cos(nt)\right)\Big|_{-\pi}^{0}=\frac{1}{n\pi}\left[-1+(-1)^n\right]=\frac{(-1)^n-1}{n\pi}$$

$$\boxed{b_n=\frac{(-1)^n-1}{n\pi}}\Rightarrow\boxed{F(t)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n\pi}\sin(nt)}$$

$$F(t)=\frac{1}{2}+\frac{(-2)}{\pi}\sin(t)+\underset{n=2}{}\frac{(-2)}{3\pi}\sin(3t)+\underset{n=4}{}\frac{(-2)}{5\pi}\sin(5t)+\cdots$$

$$\boxed{F(t)=\frac{1}{2}-\frac{2}{\pi}\sum_{n=1,3,5,7}^{\infty}\frac{\sin((2n-1)t)}{(2n-1)}}$$

$$F(t)=\frac{1}{2}-\frac{2}{\pi}\sin(t)$$

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۱۳ ص

محل مهر آموزش

$$m\ddot{x}+c\dot{x}+kx=\frac{1}{2} \Rightarrow x=A'=\text{constant} \Rightarrow \dot{x}=\ddot{x}=0$$

$$kx=\frac{1}{2} \Rightarrow \boxed{x_{p_1}=\frac{1}{2k}}$$

$$m\ddot{x}+c\dot{x}+kx=-\frac{2}{\pi}\sin(t) \Rightarrow x_{p_2}=A'_1\sin t+A'_2\cos t \Rightarrow$$

$$\dot{x}_{p_2}=A'_1\cos t-A'_2\sin t \Rightarrow \ddot{x}_{p_2}=-A'_1\sin t-A'_2\cos t \Rightarrow$$

$$m[-A'_1\sin t-A'_2\cos t]+c[A'_1\cos t-A'_2\sin t]+k[A'_1\sin t+A'_2\cos t]=-\frac{2}{\pi}\sin(t)$$

$$(-mA'_1-cA'_2+kA'_1]\sin(t)+[-mA'_2+cA'_1+kA'_2]\cos t=-\frac{2}{\pi}\sin(t)$$

$$\begin{cases}(k-m)A'_1-cA'_2=-\frac{2}{\pi}\\(k-m)A'_2+cA'_1=0\end{cases} \Rightarrow A'_2=-\frac{c}{k-m}A'_1$$

$$(k-m)A'_1-c\times\left(-\frac{c}{k-m}A'_1\right)=-\frac{2}{\pi} \Rightarrow \left[(k-m)+\frac{c^2}{(k-m)}\right]A'_1=-\frac{2}{\pi}$$

$$\left[\frac{(k-m)^2+c^2}{k-m}\right]A'_1=-\frac{2}{\pi} \Rightarrow A'_1=\frac{k-m}{(k-m)^2+c^2}\times-\frac{2}{\pi} \Rightarrow$$

$$\boxed{A'_1=-\frac{2}{\pi}\frac{k-m}{(k-m)^2+c^2}} \Rightarrow \boxed{A'_2=\frac{2}{\pi}\frac{c}{(k-m)^2+c^2}} \Rightarrow$$

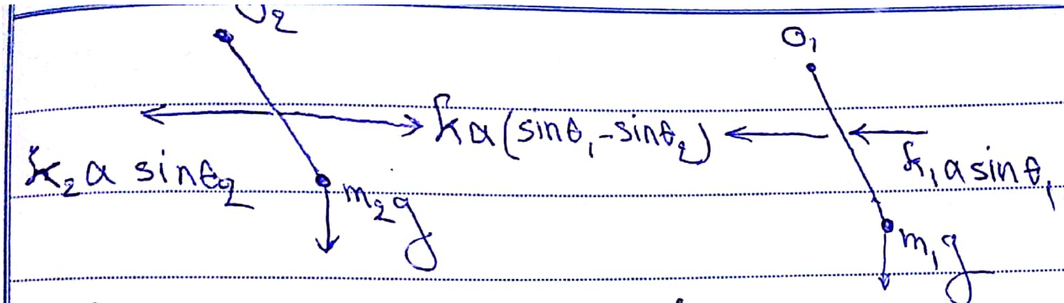
$$x_p=x_{p_1}+x_{p_2}=\frac{1}{2k}+[A'_1\sin(t)+A'_2\cos(t)] \Rightarrow x=x_h+x_p$$

$$x=e^{-t}[A_1\cos(14.10674t)+A_2\sin(14.10674t)]+\frac{1}{2\times1000}$$

$$-\frac{2}{\pi}\times\frac{1000-5}{(1000-5)^2+10^2}\sin(t)+\frac{2}{\pi}\frac{10}{(1000-5)^2+10^2}\cos(t)$$

چاپ نیما ۳۳۳۳۹۷۷ - کد ۱۷۵٤

جواب سؤال ۴



for small θ_1 and θ_2 $\Rightarrow$ $\begin{cases} \sin\theta_1 \simeq \theta_1 \\ \sin\theta_2 \simeq \theta_2 \end{cases}$ and $\begin{cases} \cos\theta_1 \simeq 1 \\ \cos\theta_2 \simeq 1 \end{cases}$

$\curvearrowleft +M$

$$\sum M)_{O_1} = I_1 \ddot{\theta}_1 \Rightarrow$$

$$\begin{cases} -k_1 a \sin\theta_1 \times a\cos\theta_1 - ka(\sin\theta_1 - \sin\theta_2) \times a\cos\theta_1 - mg \times L\sin\theta_1 = (m_1 L^2)\ddot{\theta}_1 \\ -k_1 a^2 \theta_1 - ka^2(\theta_1 - \theta_2) - mgL\theta_1 = m_1 L^2 \ddot{\theta}_1 \quad (1) \end{cases}$$

$\curvearrowleft +M$

$$\sum M)_{O_2} = I_2 \ddot{\theta} \Rightarrow$$

$$+ka(\sin\theta_1 - \sin\theta_2) \times a\cos\theta_2 - k_2 a \sin\theta_2 \times a\cos\theta_2 - mg \times L\sin\theta_2 = (m_2 L^2)\ddot{\theta}_2$$

$$\Rightarrow ka^2(\theta_1 - \theta_2) - k_2 a^2 \theta_2 - mgL\theta_2 = m_2 L^2 \ddot{\theta}_2 \quad (2)$$

$$\begin{cases} m_1 L^2 \ddot{\theta}_1 + [(k_1 + k)a^2 + mgL]\theta_1 - ka^2\theta_2 = 0 \\ m_2 L^2 \ddot{\theta}_2 - ka^2\theta_1 + [(k_2 + k)a^2 + m_2 gL]\theta_2 = 0 \end{cases}$$

$$\begin{bmatrix} m_1 L^2 & 0 \\ 0 & m_2 L^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} +[(k_1+k)a^2 + m_1 g L] & -ka^2 \\ -ka^2 & +[(k_2+k)a^2+m_2 g L] \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\theta_1(t) = \Theta_1 \cos(\omega t+\phi) \Rightarrow \dot{\theta}_1(t) = -\Theta_1 \omega \sin(\omega t+\phi) \Rightarrow \ddot{\theta}_1(t) = -\Theta_1 \omega^2 \cos(\omega t+\phi)$$

$$\theta_2(t) = \Theta_2 \cos(\omega t+\phi) \Rightarrow \dot{\theta}_2(t) = -\Theta_2 \omega \sin(\omega t+\phi) \Rightarrow \ddot{\theta}_2(t) = -\Theta_2 \omega^2 \cos(\omega t+\phi)$$

$$\begin{bmatrix} -\omega^2 m_1 L^2 + [(k_1+k)a^2+m_1 g L] & -ka^2 \\ -ka^2 & -\omega^2 m_2 L^2 + [(k_2+k)a^2+m_2 g L] \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}$$

$$m_1 m_2 L^4 \omega^4 - [(k_2+k)a^2+m_2 g L] m_1 L^2 \omega^2 - [(k_1+k)a^2+m_1 g L] m_2 L^2 \omega^2$$

$$+ [(k_1+k)a^2+m_1 g L][(k_2+k)a^2+m_2 g L] - k^2 a^4 = 0$$

$$\underbrace{(m_1 m_2 L^4)}_{A} \omega^4 - \underbrace{\left\{[(k_2+k)a^2+m_2 g L] m_1 + [(k_1+k)a^2+m_1 g L] m_2\right\} L^2}_{B} \omega^2$$

$$+ \underbrace{\left\{[(k_1+k)a^2+m_1 g L] \times [(k_2+k)a^2+m_2 g L] - k^2 a^4\right\}}_{C} = 0$$

$$\omega^2_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \Rightarrow \begin{cases} \omega_1 = 7.7214 \\ \omega_2 = 12.2319 \end{cases}$$

$$\omega = \omega_1 \Rightarrow \begin{bmatrix} +15 & -15 \\ -15 & 15 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = +1 = r_1$$

$$\omega = \omega_2 = \begin{bmatrix} -7.5 & -15 \\ -15 & -30 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = -\frac{1}{2} = r_2$$

$$\theta^{(1)}_{(t)} = \begin{Bmatrix} \Theta_1^{(1)} \cos(\omega_1 t + \phi_1) \\ +\Theta_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix}, \quad \theta^{(2)}_{(t)} = \begin{Bmatrix} \Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \\ -\frac{1}{2}\Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \end{Bmatrix}$$