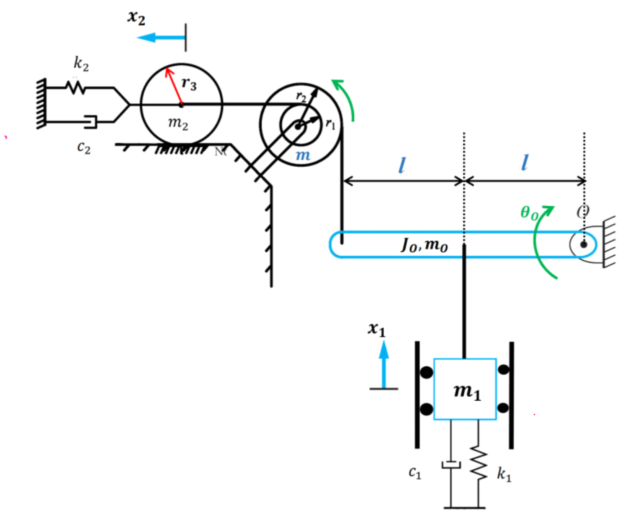
Dynamics

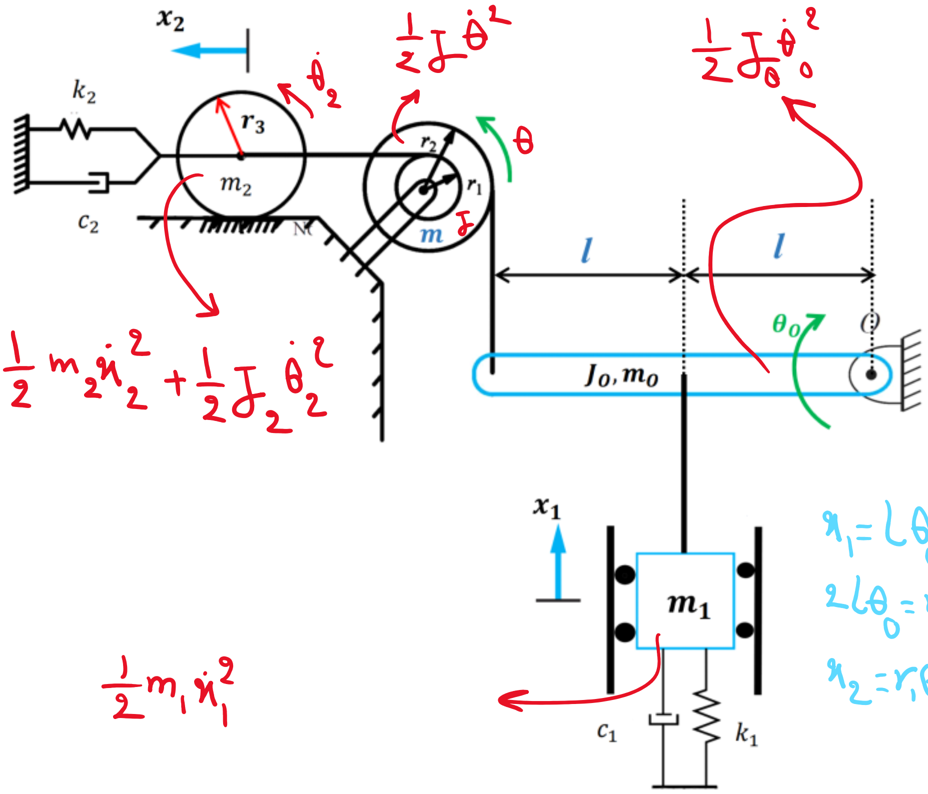
Tuesday, November 23, 2021 8:49 PM

In the system shown in the figure, calculate the equivalent mass, the equivalent spring and the equivalent damper.



k_1	$1000\ \frac{N}{m}$
k_2	$2000\ \frac{N}{m}$
c_1	$10\ \frac{N.m}{s}$
c_2	$4\ \frac{N.m}{s}$
m_1	2 kg
m_2	6 kg
m	8 kg
J_O	20 $kg.m^2$
r_1	1 m
r_2	2 m
r_3	3 m
l	10 m
x_0	-0.2 m
$\dot{x}_0$	$2\ \frac{m}{s}$

Equivalent mass



$\frac{1}{2} J \dot{\theta}^2$

$\frac{1}{2} J_0 \dot{\theta}_0^2$

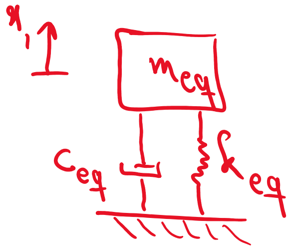
$\frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} J_2 \dot{\theta}_2^2$

$\frac{1}{2} m_1 \dot{x}_1^2$

$x_1 = l\theta_0$

$2l\theta_0 = r_2\theta$

$x_2 = r_1\theta = r_3\theta_2$



$\frac{1}{2} m_{eq} \dot{x}_1^2$

$$\frac{1}{2} m_{eq} \dot{x}_1^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} J_0 \dot{\theta}_0^2 + \frac{1}{2} J \dot{\theta}^2 + \left(\frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} J_2 \dot{\theta}_2^2 \right) \qquad (1)$$

$$\begin{cases} x_1 = L\theta_0 \Rightarrow \theta_0 = \dfrac{x_1}{L} \\ 2L\theta_0 = r_2 \theta \\ x_2 = r_1 \theta = r_3 \theta_2 \end{cases} \quad 2L\frac{x_1}{L} = r_2\theta \Rightarrow \theta = \frac{2}{r_2} x_1 \quad x_2 = r_1\left(\frac{2}{r_2} x_1\right) = 2\frac{r_1}{r_2} x_1$$

$$\theta_2 = 2\frac{r_1}{r_2 r_3} x_1$$

$$\Rightarrow \begin{cases} \theta_0 = \dfrac{x_1}{L} \\ \theta = \dfrac{2}{r_2} x_1 \\ x_2 = 2\dfrac{r_1}{r_2} x_1 \\ \theta_2 = 2\dfrac{r_1}{r_2 r_3} x_1 \end{cases} \Longrightarrow \begin{cases} \dot{\theta}_0 = \dfrac{\dot{x}_1}{L} \\ \dot{\theta} = \dfrac{2}{r_2} \dot{x}_1 \\ \dot{x}_2 = 2\dfrac{r_1}{r_2} \dot{x}_1 \\ \dot{\theta}_2 = 2\dfrac{r_1}{r_2 r_3} \dot{x}_1 \end{cases} \qquad (2)$$

(2) into (1) $\Rightarrow$

$$\frac{1}{2} m_{eq} \dot{x}_1^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} J_0 \left(\frac{\dot{x}_1}{L}\right)^2 + \frac{1}{2} J \left(\frac{2}{r_2}\dot{x}_1\right)^2 + \left[\frac{1}{2} m_2 \left(2\frac{r_1}{r_2}\dot{x}_1\right)^2 + \frac{1}{2} J_2 \left(2\frac{r_1}{r_2 r_3}\dot{x}_1\right)^2\right]$$

$$m_{eq} = m_1 + \frac{J_0}{L^2} + J\frac{4}{r^2} + 4m_2\left(\frac{r_1}{r_2}\right)^2 + 4J_2\left(\frac{r_1}{r_2 r_3}\right)^2 \qquad (3)$$

mass moment of inertia for disk $= J = \frac{1}{2} m r^2$

$$J = \frac{1}{2} m r_2^2 \quad , \quad J_2 = \frac{1}{2} m_2 r_3^2 \qquad (4)$$

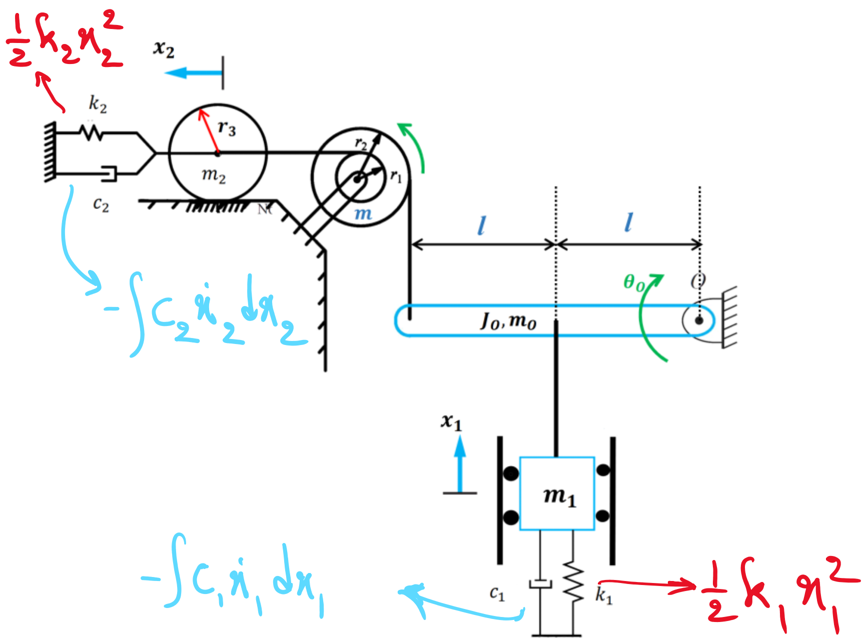
(4) into (3) $\Rightarrow m_{eq} = m_1 + \dfrac{J_0}{L^2} + \dfrac{4}{r^2}\left(\dfrac{1}{2} m r_2^2\right) + 4m_2\left(\dfrac{r_1}{r_2}\right)^2 + 4\left(\dfrac{1}{2} m_2 r_3^2\right)\left(\dfrac{r_1}{r_2 r_3}\right)^2 \Rightarrow$

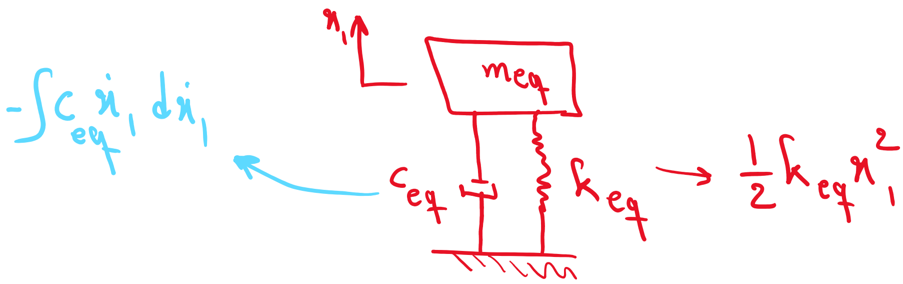
$$m_{eq} = m_1 + \frac{J_0}{L^2} + 2m + \underbrace{4m_2\left(\frac{r_1}{r_2}\right)^2 + 2m_2\left(\frac{r_1}{r_2}\right)^2}_{6m_2\left(\frac{r_1}{r_2}\right)^2} \Rightarrow$$

$$\boxed{m_{eq} = m_1 + \frac{J_0}{L^2} + 2m + 6m_2\left(\frac{r_1}{r_2}\right)^2} \qquad (5)$$

$$\boxed{m_{eq} = m_1 + \frac{J_0}{L^2} + 2m + 6m_2\left(\frac{r_1}{r_2}\right)^2} \qquad (5)$$

$k_{eq} = ?$





$$\frac{1}{2}k_{eq}x_1^2 = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 \;\rightarrow\; \underbrace{x_2 = 2\,\frac{r_1}{r_2}\,x_1} \;\Rightarrow$$

$$\frac{1}{2}k_{eq}x_1^2 = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2\left(2\,\frac{r_1}{r_2}\,x_1\right)^2 \;\Longrightarrow$$

$$\boxed{k_{eq} = k_1 + 4\left(\frac{r_1}{r_2}\right)^2 k_2} \qquad (6)$$

$$\boxed{c_{eq} = ?}$$

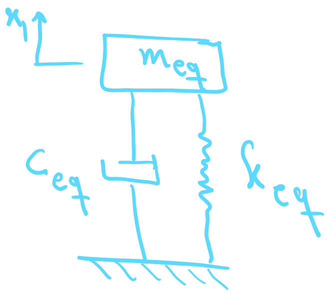
$$x_2 = 2\frac{r_1}{r_2}x_1 \Rightarrow \begin{cases} \dot{x}_2 = 2\frac{r_1}{r_2}\dot{x}_1 \\ dx_2 = 2\frac{r_1}{r_2}dx_1 \end{cases} \Rightarrow$$

$$-\int c_{eq}\dot{x}_1 dx_1 = -\int c_1 \dot{x}_1 dx_1 - \int c_2 \dot{x}_2 dx_2$$

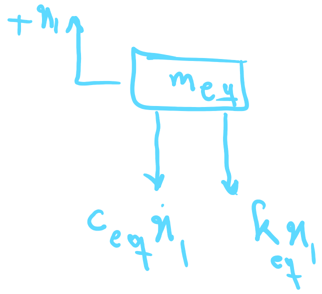
$$-\int c_{eq}\dot{x}_1 dx_1 = -\int c_1 \dot{x}_1 dx_1 - \int c_2 \left(2\frac{r_1}{r_2}\dot{x}_1\right)\left(2\frac{r_1}{r_2}dx_1\right) \Rightarrow$$

$$-\int c_{eq}\dot{x}_1 dx_1 = -\int c_1 \dot{x}_1 dx_1 - \int 4\left(\frac{r_1}{r_2}\right)^2 c_2 \dot{x}_1 dx_1 \Rightarrow$$

$$\boxed{c_{eq} = c_1 + 4\left(\frac{r_1}{r_2}\right)^2 c_2} \quad (7)$$



F.B.D.



$$\sum F_x = m_{eq}\ddot{x}_1 \Rightarrow -c_{eq}\dot{x}_1 - k_{eq}x_1 = m_{eq}\ddot{x}_1 \Rightarrow$$

$$\boxed{m_{eq}\ddot{x}_1 + c_{eq}\dot{x}_1 + k_{eq}x_1 = 0} \quad (8)$$

$$\Rightarrow \left[m_1 + \frac{J_0}{l^2} + 2m + 6m_2\left(\frac{r_1}{r_2}\right)^2\right]\ddot{x} + \left[c_1 + 4\left(\frac{r_1}{r_2}\right)^2 c_2\right]\dot{x}_1 + \left[k_1 + 4\left(\frac{r_1}{r_2}\right)^2 k_2\right]x_1 = 0$$

$$m_{eq} = 2 + \frac{20}{10^2} + 2\times 8 + 6\times 6\times\left(\frac{1}{2}\right)^2 = 36.2\ kg$$

$$c_{eq} = 10 + 4\left(\frac{1}{2}\right)^2 = 14 \ \frac{N.s}{m}$$

$$k_{eq} = 1000 + 2000\left(\frac{1}{2}\right)^2 = 3000 \ \frac{N}{m}$$

$$36.2\,\ddot{x}_1 + 14\,\dot{x}_1 + 3000\,x_1 = 0$$

$$x_1 = C_1 e^{st} \Rightarrow \dot{x}_1 = C_1 s e^{st} \Rightarrow \ddot{x} = C_1 s^2 e^{st}$$

$$36.2\, C_1 s^2 e^{st} + 14 C_1 s e^{st} + 3000\, C_1 e^{st} = 0 \Rightarrow$$

$$[36.2 s^2 + 14 s + 3000]\, C_1 e^{st} = 0 \Rightarrow 36.2 s^2 + 14 s + 3000 = 0 \Rightarrow$$

$$s_1, s_2 = \frac{-14 \pm \sqrt{14^2 - 4 \times 36.2 \times 3000}}{2 \times 36.2} \Rightarrow \begin{cases} s_1 = -0.1934 + 9.1014 i \\ s_2 = -0.1934 - 9.1014 i \end{cases} \quad \overset{x = C e^{st}}{\Longrightarrow}$$

$$x = C_1 e^{s_1 t} + C_2 e^{s_2 t} \Rightarrow x = \underbrace{C_1 e^{(-0.1934 + 9.1014 i)t} + C_2 e^{(-0.1934 - 9.1014 i)t}}$$

$$e^{\pm i\theta} = \cos\theta \pm i \sin\theta$$

$$x = C_1 \left[e^{-0.1934 t}\, e^{9.1014 i t}\right] + C_2 \left[e^{-0.1934 t}\, e^{-9.1014 i t}\right] \Rightarrow$$

$$x = e^{-0.1934 t}\left[C_1 e^{9.1014 i t} + C_2 e^{-9.1014 i t}\right] \Rightarrow$$

$$x = e^{-0.1934 t}\left[C_1(\cos 9.1014 t + i \sin 9.1014 t) + C_2(\cos 9.1014 t - i \sin 9.1014 t)\right] \Rightarrow$$

$$x = e^{-0.1934 t}\left[\underbrace{(C_1 + C_2)}_{A_1}\cos 9.1014 t + i\underbrace{(C_1 - C_2)}_{A_2}\sin 9.1014 t\right] \Rightarrow$$

$$\boxed{x = e^{-0.1934 t}\left[A_1 \cos 9.1014 t + A_2 \sin 9.1014 t\right]} \quad (9)$$

I.C 1: $x_{(0)} = 0.2$ or $x|_{t=0} = 0.2 \quad \xrightarrow{(9)}$

$$\text{I.C.1}: x_{(0)} = -0.2 \quad \text{or} \quad x\big|_{t=0} = -0.2 \xrightarrow{(9)}$$

$$x\big|_{t=0} = x_{(0)} = -0.2 = \underbrace{e^{-0.1934(0)}}_{1}\Big[\underbrace{A_1 \underbrace{\cos(9.1014(0))}_{1}}_{A_1} + \underbrace{A_2 \underbrace{\sin(9.1014(0))}_{0}}_{0}\Big] \Rightarrow$$

$$\boxed{A_1 = -0.2} \quad (10)$$

$$(9) \Rightarrow \dot{x} = -0.1934\, e^{-0.1934t}\left[A_1 \cos 9.1014t + A_2 \sin 9.1014t\right] +$$

$$e^{-0.1934t}\left[-9.1014 A_1 \sin 9.1014t + 9.1014\, A_2 \cos 9.1014t\right] \quad (11)$$

$$\text{I.C.2}: \dot{x}_{(0)} = \dot{x}\big|_{t=0} = 2$$

$$2 = -0.1934\, \overbrace{e^{-0.1934(0)}}^{1}\Big[A_1 \overbrace{\cos(9.1014(0))}^{1} + \overbrace{A_2 \overbrace{\sin(9.1014(0))}^{0}}^{0}\Big] +$$

$$9.1014 \times \underbrace{e^{-0.1934(0)}}_{1}\Big[\underbrace{-A_1 \underbrace{\sin(9.1014(0))}_{0}}_{0} + A_2 \underbrace{\cos(9.1014(0))}_{1}\Big] \Rightarrow$$

$$2 = -0.1934 A_1 + 9.1014 A_2 \Rightarrow A_2 = \frac{2 + 0.1934 A_1}{9.1014} = \frac{2 + 0.1934 \times (-0.2)}{9.1014}$$

$$\Rightarrow \boxed{A_2 = 0.2155} \quad (12)$$

$$(9), (10) \text{ and } (12) \Rightarrow \boxed{x_{(t)} = e^{-0.1934t}\left[-0.2 \cos 9.1014t + 0.2155 \sin 9.1014t\right]}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{3000}{36.2}} = 9.1035$$

$$c_c = 2\sqrt{k_{eq}\, m_{eq}} = 2\sqrt{3000 \times 36.2} = 659.0903 \ \frac{N.s}{m}$$

$$\xi = \frac{c_{eq}}{c_c} = \frac{14}{659.0903} = 0.0212 < 1 \quad \text{under damped}$$

$$\omega_d = \omega_n \sqrt{1 - \xi^2} = 9.1035 \sqrt{1 - 0.0212^2} = 9.1014$$

$$\xi \omega_n = 0.0212 \times 9.1035 = 0.1934$$