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2.6 Linear Operators

In calculus we consider the real line **R** and real-valued functions on **R** (or on a subset of **R**). Obviously, any such function is a mapping[5] of its domain into **R**. In functional analysis we consider more general spaces, such as metric spaces and normed spaces, and mappings of these spaces.

In the case of vector spaces and, in particular, normed spaces, a mapping is called an **operator**.

2.6-1 Definition (Linear operator). A *linear operator* T is an operator such that

(*i*) the domain $\mathfrak{D}(T)$ of T is a vector space and the range $\mathfrak{R}(T)$ lies in a vector space over the same field,

(ii) for all $x, y \in \mathfrak{D}(T)$ and scalars α,

$$T(x+y) = Tx + Ty$$

(1)

$$T(\alpha x) = \alpha Tx.$$

we write Tx instead of $T(x)$

$\mathfrak{D}(T)$ denotes the domain of T.
$\mathfrak{R}(T)$ denotes the range of T.
$\mathcal{N}(T)$ denotes the null space of T.

By definition, the **null space** of T is the set of all $x \in \mathfrak{D}(T)$ such that $Tx = 0$. (Another word for null space is "kernel." We shall not adopt

We should also say something about the use of arrows in connection with operators. Let $\mathscr{D}(T) \subset X$ and $\mathscr{R}(T) \subset Y$, where X and Y are vector spaces, both real or both complex. Then T is an operator *from* (or mapping *of*) $\mathscr{D}(T)$ **onto** $\mathscr{R}(T)$, written

$$T\colon\ \mathscr{D}(T) \longrightarrow \mathscr{R}(T),$$

or from $\mathscr{D}(T)$ *into* Y, written

$$T\colon\ \mathscr{D}(T) \longrightarrow Y.$$

If $\mathscr{D}(T)$ is the whole space X, then—and only then—we write

$$T\colon\ X \longrightarrow Y.$$

Clearly, (1) is equivalent to

(2) $$T(\alpha x + \beta y) = \alpha Tx + \beta Ty.$$

By taking $\alpha = 0$ in (1) we obtain the following formula which we shall need many times in our further work:

(3) $$T0 = 0.$$

Examples

2.6-2 Identity operator. The *identity operator* $I_X: X \longrightarrow X$ is defined by $I_X x = x$ for all $x \in X$. We also write simply I for I_X; thus, $Ix = x$.

2.6-3 Zero operator. The *zero operator* $0: X \longrightarrow Y$ is defined by $0x = 0$ for all $x \in X$.

2.6-4 Differentiation. Let X be the vector space of all polynomials on $[a, b]$. We may define a linear operator T on X by setting

$$Tx(t) = x'(t)$$

for every $x \in X$, where the prime denotes differentiation with respect to t. This operator T maps X onto itself.

2.6-5 Integration. A linear operator T from $C[a, b]$ into itself can be defined by

$$Tx(t) = \int_a^t x(\tau)\, d\tau \qquad t \in [a, b].$$

2.6-6 Multiplication by t. Another linear operator from $C[a, b]$ into itself is defined by

$$Tx(t) = tx(t).$$

2.6-7 Elementary vector algebra. The *cross product* with one factor kept fixed defines a linear operator T_1: $\mathbf{R}^3 \longrightarrow \mathbf{R}^3$. Similarly, the *dot product* with one fixed factor defines a linear operator T_2: $\mathbf{R}^3 \longrightarrow \mathbf{R}$, say,

$$T_2 x = x \cdot a = \xi_1\alpha_1 + \xi_2\alpha_2 + \xi_3\alpha_3$$

where $a = (\alpha_j) \in \mathbf{R}^3$ is fixed.

2.6-8 Matrices. A *real matrix* $A = (\alpha_{jk})$ with r rows and n columns defines an operator T: $\mathbf{R}^n \longrightarrow \mathbf{R}^r$ by means of

$$y = Ax$$

where $x = (\xi_j)$ has n components and $y = (\eta_j)$ has r components and both vectors are written as column vectors because of the usual convention of matrix multiplication; writing $y = Ax$ out, we have

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \cdot \\ \cdot \\ \cdot \\ \eta_r \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \cdot \\ \xi_n \end{bmatrix}.$$

T is linear because matrix multiplication is a linear operation. If A were complex, it would define a linear operator from $\mathbf{C}^n$ into $\mathbf{C}^r$. A detailed discussion of the role of matrices in connection with linear operators follows in Sec. 2.9. ∎

2.6-9 Theorem (Range and null space). *Let T be a linear operator. Then:*

(a) *The range $\mathscr{R}(T)$ is a vector space.*

(b) *If $\dim \mathscr{D}(T) = n < \infty$, then $\dim \mathscr{R}(T) \leqq n$.*

(c) *The null space $\mathscr{N}(T)$ is a vector space.*

Proof. **(a)** We take any $y_1, y_2 \in \mathscr{R}(T)$ and show that $\alpha y_1 + \beta y_2 \in \mathscr{R}(T)$ for any scalars α, β. Since $y_1, y_2 \in \mathscr{R}(T)$, we have $y_1 = Tx_1$, $y_2 = Tx_2$ for some $x_1, x_2 \in \mathscr{D}(T)$, and $\alpha x_1 + \beta x_2 \in \mathscr{D}(T)$ because $\mathscr{D}(T)$ is a vector space. The linearity of T yields

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = \alpha y_1 + \beta y_2.$$

Hence $\alpha y_1 + \beta y_2 \in \mathscr{R}(T)$. Since $y_1, y_2 \in \mathscr{R}(T)$ were arbitrary and so were the scalars, this proves that $\mathscr{R}(T)$ is a vector space.

(b) We choose $n+1$ elements $y_1, \cdots, y_{n+1}$ of $\mathcal{R}(T)$ in an arbitrary fashion. Then we have $y_1 = Tx_1, \cdots, y_{n+1} = Tx_{n+1}$ for some $x_1, \cdots, x_{n+1}$ in $\mathcal{D}(T)$. Since $\dim \mathcal{D}(T) = n$, this set $\{x_1, \cdots, x_{n+1}\}$ must be linearly dependent. Hence

$$\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1} = 0$$

for some scalars $\alpha_1, \cdots, \alpha_{n+1}$, not all zero. Since T is linear and $T0 = 0$, application of T on both sides gives

$$T(\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1}) = \alpha_1 y_1 + \cdots + \alpha_{n+1} y_{n+1} = 0.$$

This shows that $\{y_1, \cdots, y_{n+1}\}$ is a linearly dependent set because the α_j's are not all zero. Remembering that this subset of $\mathcal{R}(T)$ was chosen in an arbitrary fashion, we conclude that $\mathcal{R}(T)$ has no linearly independent subsets of $n+1$ or more elements. By the definition this means that $\dim \mathcal{R}(T) \leqq n$.

(c) We take any $x_1, x_2 \in \mathcal{N}(T)$. Then $Tx_1 = Tx_2 = 0$. Since T is linear, for any scalars α, β we have

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = 0.$$

This shows that $\alpha x_1 + \beta x_2 \in \mathcal{N}(T)$. Hence $\mathcal{N}(T)$ is a vector space. ∎

An immediate consequence of part (b) of the proof is worth noting:

Linear operators preserve linear dependence.

Let us turn to the inverse of a linear operator. We first remember that a mapping $T\colon \mathscr{D}(T) \longrightarrow Y$ is said to be **injective** or **one-to-one** if different points in the domain have different images, that is, if for any $x_1, x_2 \in \mathscr{D}(T)$,

$$(4) \qquad x_1 \neq x_2 \quad \Longrightarrow \quad Tx_1 \neq Tx_2;$$

equivalently,

$$(4^*) \qquad Tx_1 = Tx_2 \quad \Longrightarrow \quad x_1 = x_2.$$

In this case there exists the mapping

$$(5) \qquad \begin{aligned} T^{-1}\colon \mathscr{R}(T) &\longrightarrow \mathscr{D}(T) \\ y_0 &\longmapsto x_0 \qquad (y_0 = Tx_0) \end{aligned}$$

which maps every $y_0 \in \mathscr{R}(T)$ onto that $x_0 \in \mathscr{D}(T)$ for which $Tx_0 = y_0$. See Fig. 20. The mapping T^{-1} is called the **inverse**[6] of T.

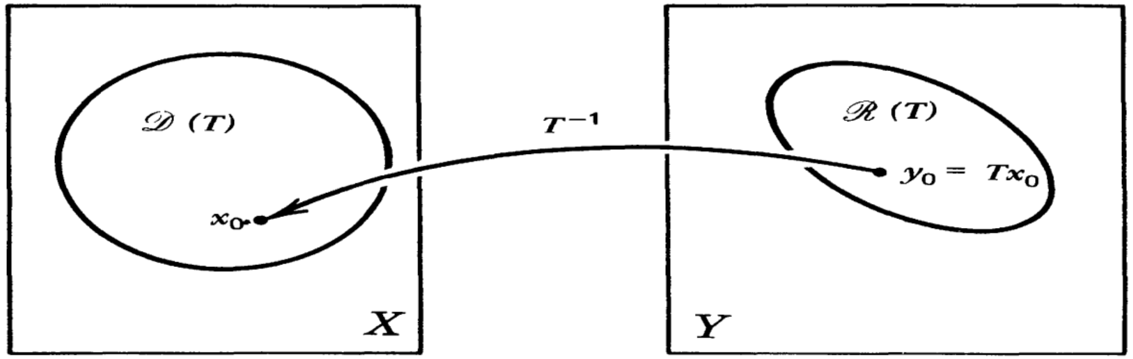

Fig. 20. Notations in connection with the inverse of a mapping; cf. (5)

$$T^{-1}Tx = x \qquad \text{for all } x \in \mathcal{D}(T)$$

$$TT^{-1}y = y \qquad \text{for all } y \in \mathcal{R}(T).$$

2.6-10 Theorem (Inverse operator). *Let X, Y be vector spaces, both real or both complex. Let $T\colon \mathcal{D}(T) \longrightarrow Y$ be a linear operator with domain $\mathcal{D}(T) \subset X$ and range $\mathcal{R}(T) \subset Y$. Then:*

(a) *The inverse $T^{-1}\colon \mathcal{R}(T) \longrightarrow \mathcal{D}(T)$ exists if and only if*

$$Tx = 0 \qquad \Longrightarrow \qquad x = 0.$$

(b) *If T^{-1} exists, it is a linear operator.*

(c) *If $\dim \mathcal{D}(T) = n < \infty$ and T^{-1} exists, then $\dim \mathcal{R}(T) = \dim \mathcal{D}(T)$.*

Proof. **(a)** Suppose that $Tx = 0$ implies $x = 0$. Let $Tx_1 = Tx_2$. Since T is linear,

$$T(x_1 - x_2) = Tx_1 - Tx_2 = 0,$$

so that $x_1 - x_2 = 0$ by the hypothesis. Hence $Tx_1 = Tx_2$ implies $x_1 = x_2$, and T^{-1} exists by (4*). Conversely, if T^{-1} exists, then (4*) holds. From (4*) with $x_2 = 0$ and (3) we obtain

$$Tx_1 = T0 = 0 \qquad \Longrightarrow \qquad x_1 = 0.$$

This completes the proof of (a).

(b) We assume that T^{-1} exists and show that T^{-1} is linear. The domain of T^{-1} is $\mathcal{R}(T)$ and is a vector space by Theorem 2.6-9(a). We consider any $x_1, x_2 \in \mathcal{D}(T)$ and their images

$$y_1 = Tx_1 \qquad \text{and} \qquad y_2 = Tx_2.$$

Then

$$x_1 = T^{-1}y_1 \qquad \text{and} \qquad x_2 = T^{-1}y_2.$$

T is linear, so that for any scalars α and β we have

$$\alpha y_1 + \beta y_2 = \alpha T x_1 + \beta T x_2 = T(\alpha x_1 + \beta x_2).$$

Since $x_j = T^{-1}y_j$, this implies

$$T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2 = \alpha T^{-1}y_1 + \beta T^{-1}y_2$$

and proves that T^{-1} is linear.

(c) We have $\dim \mathscr{R}(T) \leqq \dim \mathscr{D}(T)$ by Theorem 2.6-9(*b*), and $\dim \mathscr{D}(T) \leqq \dim \mathscr{R}(T)$ by the same theorem applied to T^{-1}. ∎

2.6-11 Lemma (Inverse of product). *Let $T\colon X \longrightarrow Y$ and $S\colon Y \longrightarrow Z$ be bijective linear operators, where X, Y, Z are vector spaces (see* Fig. 21). *Then the inverse $(ST)^{-1}\colon Z \longrightarrow X$ of the product (the composite) ST exists, and*

$$(ST)^{-1} = T^{-1}S^{-1}. \tag{6}$$

Proof. The operator $ST\colon X \longrightarrow Z$ is bijective, so that $(ST)^{-1}$ exists. We thus have

$$ST(ST)^{-1} = I_Z$$

where I_Z is the identity operator on Z. Applying S^{-1} and using $S^{-1}S = I_Y$ (the identity operator on Y), we obtain

$$S^{-1}ST(ST)^{-1} = T(ST)^{-1} = S^{-1}I_Z = S^{-1}.$$

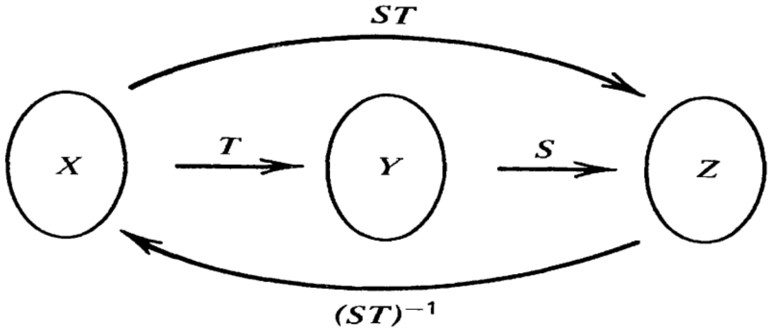

Fig. 21. Notations in Lemma 2.6-11

Applying T^{-1} and using $T^{-1}T = I_X$, we obtain the desired result

$$T^{-1}T(ST)^{-1} = (ST)^{-1} = T^{-1}S^{-1}.$$

This completes the proof. ∎

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با تشکر