بسم الله الرحمن الرحیم

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2.5 Compactness and Finite Dimension

A few other basic properties of finite dimensional normed spaces and subspaces are related to the concept of compactness. The latter is defined as follows.

2.5-1 Definition (Compactness). A metric space X is said to be *compact*[4] if every sequence in X has a convergent subsequence. A subset M of X is said to be *compact* if M is compact considered as a subspace of X, that is, if every sequence in M has a convergent subsequence whose limit is an element of M. ∎

A general property of compact sets is expressed in

2.5-2 Lemma (Compactness). *A compact subset M of a metric space is closed and bounded.*

proof. Let $x \in \overline{M} \xrightarrow{\text{by Lemma 1.4-6(a)}} \exists \{x_n\} \subset M$ s.t. $x_n \longrightarrow x$

Since $\{x_n\} \subset M$ and M is compact $\Rightarrow \exists$ a subseq. $\{x_{n_k}\}$ of $\{x_n\} \ni x_{n_k} \longrightarrow x'$ (Note that $x' \in M$).

since $\{x_{n_k}\}$ is a subseq. of $\{x_n\}$ & $x_n \longrightarrow x$

$\Longrightarrow x_{n_k} \longrightarrow x$. So, $x = x' \Longrightarrow x \in M$ / $\therefore \overline{M} \subset M$

since $M \subset \overline{M} \Longrightarrow \overline{M} = M$ i.e. M is closed.

Now, we prove that M is bounded. If M were unbounded $\Longrightarrow \sup\limits_{(x,y)\in M} d(x,y) \not< +\infty$.

$\Longrightarrow \forall n \in \mathbb{N} \; \exists \; y_n \in M \ni d(y_n, b) > n$, where b is any fixed element of M. So, sequence $\{y_n\}$ and any subsequence of $\{y_n\}$ is unbounded.

$\Longrightarrow \{y_n\}$ could not have a convergent subsequence (since a convergent subsequence must be bounded, by Lemma 1.4-2.)

$\Longrightarrow M$ is not compact $\#$ $\therefore$ M is bounded.

The converse of this lemma is in general false.

Example: Let $X = \ell^2$ & $M = \{(e_n) : e_n = (\delta_{nj})\}$
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$e_1 = (1, 0, 0, \cdots \text{———})$
$e_2 = (0, 1, 0, 0, \cdots \text{———})$
$\vdots$
مولفه n ام
$e_n = (0, 0, \cdots \text{———}, 0, 1, 0, 0, \cdots \text{———})$
$\vdots$

M is bounded, because $\|e_n\|_2 = \sum_{i=1}^{\infty} |(e_n)_i|^2 = 1.$
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since $\|e_n - e_m\|_2 = \sum_{i=1}^{\infty} |(e_n)_i - (e_m)_i|^2 = 2, \forall n \neq m$

$\Rightarrow$ M has no poit of accumulation $\Rightarrow$ M is closed.

But M is not compact (since $\|e_n - e_m\|_2 = 2, \forall n \neq m$
$\Rightarrow \{e_n\}$ could not have a convergent subsequence)

However, for a finite dimensional normed space we have

2.5-3 Theorem (Compactness). *In a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded.*

proof. If M is compact $\xRightarrow{\text{by Lemma 2.5-2}}$ M is closed and bounded.

For conversely, Let M be closed and bounded.

We show that M is compact. Let (x_m) be any arbitrary sequence in M. Let $\dim X = n$ and $\{e_1, \dots, e_n\}$ a basis for X. So, $x_m = \xi_1^{(m)} e_1 + \xi_2^{(m)} + \dots + \xi_n^{(m)} e_n$.

Since M is bounded, then (x_m) is bounded since M is bounded, then (x_m) is bounded

$\Rightarrow \exists\, k>0 \ni \|x_m\| \le k,\ \forall m \xrightarrow{\text{by Lemma 2.4-1}} \exists\ c>0$ s.t.

$$k \ge \|x_m\| = \|\xi_1^{(m)} e_1 + \cdots + \xi_n^{(m)} e_n\| \ge c \sum_{j=1}^{n} |\xi_j^{(m)}|$$

$$\Longrightarrow \sum_{j=1}^{n} |\xi_j^{(m)}| \le \frac{k}{c} \Rightarrow |\xi_j^{(m)}| \le \frac{k}{c},\ \forall j, m$$

$\Longrightarrow$ for each fixed j the sequence of numbers

$(\xi_j^{(m)})$ is bounded $\xrightarrow[\text{الاثالبرز}]{\text{بنابر قضیه ای از}}$ has a convergent

subsequene with limit of ξ_j.

As in the proof of Lemma 2.4-1 we conclude that

(x_m) has a subsequence (y_m) which convergent

to $z = \sum_{j=1}^{n} \xi_j e_j$. since $(y_m) \subset M$, $y_m \longrightarrow z$ and M

is closed $\Longrightarrow z \in M$. This shows that the arbitrary

sequence (x_m) in M has a subsequence which converges in M.

$\Rightarrow M$ is compact. $\blacksquare$

2.5-4 F. Riesz's Lemma. *Let Y and Z be subspaces of a normed space X (of any dimension), and suppose that Y is closed and is a proper subset of Z. Then for every real number θ in the interval $(0, 1)$ there is a $z \in Z$ such that*

$$\|z\| = 1, \qquad \|z - y\| \geqq \theta \text{ for all } y \in Y.$$

Proof. Let $v \in Z - Y$ be arbitrary. put

$$a := \inf_{y \in Y} \|v - y\| \longrightarrow$$ a برابر با فاصله v تا Y

Y is closed

since $v \notin Y \overset{\uparrow}{=} \overline{Y} \Rightarrow v \notin \overline{Y} \Longrightarrow a > 0$. Now, we take

$0 < \theta < 1$. Since $a < \frac{a}{\theta}$, $0 < a = \inf\limits_{y \in Y} \|v - y\| \xrightarrow[\text{infimum}]{\text{by definition of}}$

چون $0<\theta<1$ و $a>0$

$$\exists\, y_0 \in Y \ni 0 < a \le \|v - y_0\| \le \frac{a}{\theta} \quad (1)$$

Let $\quad c := \dfrac{1}{\underbrace{\|v-y_0\|}_{>0}}$ & $z := c(v-y_0)$.

$$\Longrightarrow \|z\| = \|c(v-y_0)\| = |c|\,\|v-y_0\| = c\|v-y_0\|$$

$$= \frac{\|v-y_0\|}{\|v-y_0\|} = 1 \Longrightarrow \|z\| = 1.$$

Now, we show that $\|z-y\| \geq \theta, \forall y \in Y$.

consider $\|z-y\| = \|c(v-y_0)-y\| = \|c(v-y_0-c^{-1}y)\|$

$$\overset{c>0}{=} c\|v-y_0-c^{-1}y\| = c\|v-(\underbrace{y_0+c^{-1}y}_{y_1 \in Y})\|$$ (Y یک زیرفضا)

$$= c\|v-y_1\| \quad ۲$$

$$\xrightarrow[a=\inf_{y\in Y}\|v-y\|]{\text{since } y_1 \in Y} \boxed{\|v-y_1\| \geq a} \quad ۳$$

$$\overset{\text{by}(2)}{\Longrightarrow} \|z-y\| = c\|v-y_1\| \underset{c>0}{\overset{\text{by}(3)}{\geq}} ca = \frac{a}{\|v-y_0\|} \overset{\text{by}(1)}{\geq} \frac{a}{\frac{a}{\theta}} = \theta.$$

$\Longrightarrow \|z-y\| \geq \theta.$ Since $y \in Y$ was arbitrary, this completes the proof. $\blacksquare$

توجه: در فضاهای نرم دار با بعد متناهی X هر گوی واحد بسته یعنی $ballX = \{x \in X ; \|x\| \leq 1\}$ فشرده است. زیرا $ballX$ بسته و کراندار می باشد لذا بنابر قضیه 2-5-3 فشرده می باشد.

* با استفاده از لم ریز نشان می دهیم که عکس این مطلب نیز درست می باشد، یعنی اگر در یک فضای نرم دار گوی واحد بسته فشرده باشد، آنگاه X دارای بعد متناهی است.

توجه: در فضاهای با بُعد نامتناهی لزوماً گوی واحد بسته فشرده نمی باشد. برای مثال، فرض کنید $X=\ell^2$. چون $\|e_n\|_2=1$ به ازای هر $n\in\mathbb{N}$. پس $e_n\in ballX$ به ازای هر $n\in\mathbb{N}$. لذا

$$(e_n)\subset ballX$$

اما این دنباله دارای زیر دنباله همگرا نیست (زیرا $\|e_n-e_m\|_2=2\not\to 0$).

پس $ballX$ فشرده نیست.

2.5-5 Theorem (Finite dimension). *If a normed space X has the property that the closed unit ball $M = \{x \mid \|x\| \leqq 1\}$ is compact, then X is finite dimensional.*

Proof. We assume that M is compact but $\dim X = \infty$, and show that this leads to a contradiction. We choose any x_1 of norm 1. This x_1 generates a one dimensional subspace X_1 of X, which is closed (cf. 2.4-3) and is a proper subspace of X since $\dim X = \infty$. By Riesz's lemma there is an $x_2 \in X$ of norm 1 such that

$$\|x_2 - x_1\| \geqq \theta = \frac{1}{2}.$$

The elements x_1, x_2 generate a two dimensional proper closed subspace X_2 of X. By Riesz's lemma there is an x_3 of norm 1 such that for all $x \in X_2$ we have

$$\|x_3 - x\| \geqq \frac{1}{2}.$$

In particular,

$$\|x_3 - x_1\| \geqq \frac{1}{2},$$

$$\|x_3 - x_2\| \geqq \frac{1}{2}.$$

Proceeding by induction, we obtain a sequence (x_n) of elements $x_n \in M$ such that

$$\|x_m - x_n\| \geqq \frac{1}{2} \qquad (m \neq n).$$

Obviously, (x_n) cannot have a convergent subsequence. This contradicts the compactness of M. Hence our assumption $\dim X = \infty$ is false, and $\dim X < \infty$. ∎

2.5-6 Theorem (Continuous mapping). *Let X and Y be metric spaces and $T: X \longrightarrow Y$ a continuous mapping (cf. 1.3-3). Then the image of a compact subset M of X under T is compact.*

proof. we show that $T(M)$ is compact. Let (y_n) be a sequence arbitrary in $T(M)$.

$\Longrightarrow \forall n;\ y_n \in T(M) \Longrightarrow y_n = Tx_n$ for some $x_n \in M$

$\Longrightarrow (x_n) \subset M$, since M is compact $\Longrightarrow$

$\exists$ a subsequence (x_{n_k}) of $(x_n) \ni x_{n_k} \longrightarrow x \in M$.

Let $y_{n_k} := Tx_{n_k} \Rightarrow (y_{n_k})$ is a subseq. of (y_n)

since $x_{n_k} \longrightarrow x$ and T is cont. $\xRightarrow{\text{by } 1.4\text{-}8}$

$Tx_{n_k} \longrightarrow Tx \Longrightarrow y_{n_k} \longrightarrow Tx \in T(M)$

so, $\{y_{n_k}\}$ is a subseq. of $\{y_n\}$ which converges in $T(M)$. Hence $T(M)$ is compact. $\blacksquare$

نتیجه ۷-۲۰۵ (ماکزیمم و می‌نیمم): فرض کنید X یک فضای متریک و M یک زیرمجموعه‌ی فشرده از X باشد. اگر $T: X \longrightarrow \mathbb{R}$ نگاشتی پیوسته باشد، در اینصورت نگاشت T به ماکزیمم و می‌نیمم مقدار خود را در بعضی از نقاط M می‌رسد.

اثبات: چون T پیوسته و M فشرده است، لذا بنابر قضیه قبل $T(M) \subset \mathbb{R}$ فشرده می‌باشد. در نتیجه بنابر لم ۲-۲۰۵، $T(M)$ بسته و کراندار است. از آنجایی که $T(M) \subset \mathbb{R}$ مجموعه‌ای کراندار است، پس $\sup T(M)$ و $\inf T(M)$ وجود دارند

و از آنجایی که $T(M)$ نسبتاً است، لذا $\sup T(M) \in T(M)$ و

$\inf T(M) \in T(M)$.

$$\inf T(M) \in T(M) \Rightarrow \exists\, \underline{x \in M} \text{ s.t. } \inf T(M) = T(x)$$

$$\Rightarrow T(x) \leq T(y),\ \forall y \in M$$

می نیمم مقدار T روی M.

$$\sup T(M) \in T(M) \Rightarrow \exists\, x' \in M \text{ s.t. } \sup T(M) = T(x')$$

$$\longrightarrow T(x') \geq T(y),\ \forall y \in M$$

ماکزیمم مقدار T روی M.

پایان

با تشکر