SOLUTIONS MANUAL

for

Extra Homework Problems from Companion Website Discrete-Time Signal Processing, 3e

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Solutions – Chapter 2

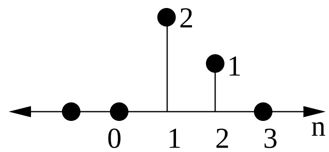
Discrete-Time Signals and Systems

2.1. We use the graphical approach to compute the convolution:

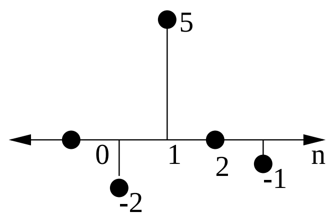
$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \end{aligned}$$

(a) $y[n] = x[n] * h[n]$

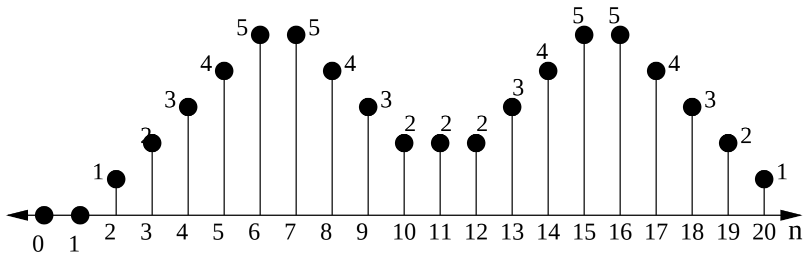
$$y[n] = \delta[n-1] * h[n] = h[n-1]$$



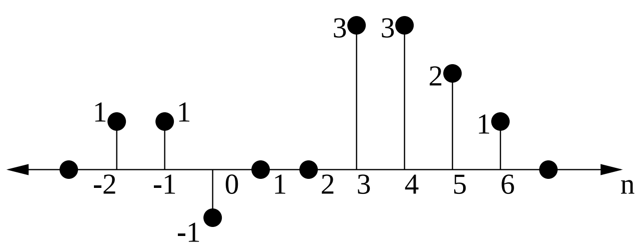
(b) $y[n] = x[n] * h[n]$



(c) $y[n] = x[n] * h[n]$



(d) $y[n] = x[n] * h[n]$



2.2. The response of the system to a delayed step:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} u[k-4]h[n-k] \end{aligned}$$

$$y[n] = \sum_{k=4}^{\infty} h[n-k]$$

Evaluating the above summation:

$$\begin{array}{ll}
\text{For } n < 4: & y[n] = 0 \\
\text{For } n = 4: & y[n] = h[0] = 1 \\
\text{For } n = 5: & y[n] = h[1] + h[0] = 2 \\
\text{For } n = 6: & y[n] = h[2] + h[1] + h[0] = 3 \\
\text{For } n = 7: & y[n] = h[3] + h[2] + h[1] + h[0] = 4 \\
\text{For } n = 8: & y[n] = h[4] + h[3] + h[2] + h[1] + h[0] = 2 \\
\text{For } n \geq 9: & y[n] = h[5] + h[4] + h[3] + h[2] + h[1] + h[0] = 0
\end{array}$$

2.3. The output is obtained from the convolution sum:

$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= \sum_{k=-\infty}^{\infty} x[k]u[n-k]
\end{aligned}$$

The convolution may be broken into five regions over the range of n:

$$y[n] = 0, \text{for } n < 0$$

$$\begin{aligned}
y[n] &= \sum_{k=0}^{n} a^k \\
&= \frac{1-a^{(n+1)}}{1-a}, \text{for } 0 \leq n \leq N_1
\end{aligned}$$

$$\begin{aligned}
y[n] &= \sum_{k=0}^{N_1} a^k \\
&= \frac{1-a^{(N_1+1)}}{1-a}, \text{ for } N_1 < n < N_2
\end{aligned}$$

$$\begin{aligned}
y[n] &= \sum_{k=0}^{N_1} a^k + \sum_{k=N_2}^{n} a^{(k-N_2)} \\
&= \frac{1-a^{(N_1+1)}}{1-a} + \frac{1-a^{(n+1)}}{1-a} \\
&= \frac{2-a^{(N_1+1)}-a^{(n+1)}}{1-a}, \text{ for } N_2 \leq n \leq (N_1+N_2)
\end{aligned}$$

$$\begin{aligned}
y[n] &= \sum_{k=0}^{N_1} a^k + \sum_{k=N_2}^{N_1+N_2} a^{(k-N_2)} \\
&= \sum_{k=0}^{N_1} a^k + \sum_{m=0} N_1 a^m
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{k=0}^{N_1} a^k \\
&= 2\cdot\left(\frac{1-a^{(N_1+1)}}{1-a}\right), \text{ for } n > (N_1+N_2)
\end{aligned}
$$

2.4. Recall that an eigenfunction of a system is an input signal which appears at the output of the system scaled by a complex constant.

(a) $x[n] = 5^n u[n]$:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]5^{(n-k)}u[n-k] \\
&= 5^n \sum_{k=-\infty}^{n} h[k]5^{-k}
\end{aligned}
$$

Becuase the summation depends on n, $x[n]$ is NOT AN EIGENFUNCTION.

(b) $x[n] = e^{j2\omega n}$:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]e^{j2\omega(n-k)} \\
&= e^{j2\omega n}\sum_{k=-\infty}^{\infty} h[k]e^{-j2\omega k} \\
&= e^{j2\omega n}\cdot H(e^{j2\omega})
\end{aligned}
$$

YES, EIGENFUNCTION.

(c) $e^{j\omega n} + e^{j2\omega n}$:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} + \sum_{k=-\infty}^{\infty} h[k]e^{j2\omega(n-k)} \\
&= e^{j\omega n}\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} + e^{j2\omega n}\sum_{k=-\infty}^{\infty} h[k]e^{-j2\omega k} \\
&= e^{j\omega n}\cdot H(e^{j\omega}) + e^{j2\omega n}\cdot H(e^{j2\omega})
\end{aligned}
$$

Since the input cannot be extracted from the above expression, the sum of complex exponentials is NOT AN EIGENFUNCTION. (Although, separately the inputs are eigenfunctions. In general, complex exponential signals are always eigenfunctions of LTI systems.)

(d) $x[n] = 5^n$:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]5^{(n-k)} \\
&= 5^n \sum_{k=-\infty}^{\infty} h[k]5^{-k}
\end{aligned}
$$

YES, EIGENFUNCTION.

(e) $x[n] = 5^n e^{j2\omega n}$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k] 5^{(n-k)} e^{j2\omega(n-k)} \\ &= 5^n e^{j2\omega n} \sum_{k=-\infty}^{\infty} h[k] 5^{-k} e^{-j2\omega k} \end{aligned}$$

YES, EIGENFUNCTION.

2.5.

- System A:

$$x[n] = (\frac{1}{2})^n$$

This input is an eigenfunction of an LTI system. That is, if the system is linear, the output will be a replica of the input, scaled by a complex constant.
Since $y[n] = (\frac{1}{4})^n$, System A is NOT LTI.

- System B:

$$x[n] = e^{jn/8} u[n]$$

The Fourier transform of $x[n]$ is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} e^{jn/8} u[n] e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} e^{-j(\omega - \frac{1}{8})n} \\ &= \frac{1}{1 - e^{-j(\omega - \frac{1}{8})}}. \end{aligned}$$

The output is $y[n] = 2x[n]$, thus

$$Y(e^{j\omega}) = \frac{2}{1 - e^{-j(\omega - \frac{1}{8})}}.$$

Therefore, the frequency response of the system is

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= 2. \end{aligned}$$

Hence, the system is a linear amplifier. We conclude that System B is LTI, and unique.

- System C: Since $x[n] = e^{jn/8}$ is an eigenfunction of an LTI system, we would expect the output to be given by

$$y[n] = \gamma e^{jn/8},$$

where γ is some complex constant, if System C were indeed LTI. The given output, $y[n] = 2e^{jn/8}$, indicates that this is so.
Hence, System C is LTI. However, it is not unique, since the only constraint is that

$$H(e^{j\omega})|_{\omega=1/8} = 2.$$

2.6. (a) The homogeneous solution $y_h[n]$ solves the difference equation when $x[n] = 0$. It is in the form $y_h[n] = \sum A(c)^n$, where the c's solve the quadratic equation

$$c^2 + \frac{1}{15}c - \frac{2}{5} = 0$$

So for $c = 1/3$ and $c = -2/5$, the general form for the homogeneous solution is:

$$y_h[n] = A_1(\frac{1}{3})^n + A_2(-\frac{2}{5})^n$$

(b) We use the z-transform, and use different ROCs to generate the causal and anti-causal impulses responses:

$$\begin{aligned} H(z) &= \frac{1}{(1-\frac{1}{3}z^{-1})(1+\frac{2}{5}z^{-1})} = \frac{5/11}{1-\frac{1}{3}z^{-1}} + \frac{6/11}{1+\frac{2}{5}z^{-1}} \\ h_c[n] &= \frac{5}{11}(\frac{1}{3})^n u[n] + \frac{6}{11}(-\frac{2}{5})^n u[n] \\ h_{ac}[n] &= -\frac{5}{11}(\frac{1}{3})^n u[-n-1] - \frac{6}{11}(-\frac{2}{5})^n u[-n-1] \end{aligned}$$

(c) Since $h_c[n]$ is causal, and the two exponential bases in $h_c[n]$ are both less than 1, it is absolutely summable. $h_{ac}[n]$ grows without bounds as n approaches $-\infty$.

(d)

$$\begin{aligned} Y(z) &= X(z)H(z) \\ &= \frac{1}{1-\frac{3}{5}z^{-1}} \cdot \frac{1}{(1-\frac{1}{3}z^{-1})(1+\frac{2}{5}z^{-1})} \\ &= \frac{-25/44}{1-1/3z^{-1}} + \frac{55/12}{1+2/5z^{-1}} + \frac{27/20}{1-3/5z^{-1}} \\ y[n] &= \frac{-25}{44}(\frac{1}{3})^n u[n] + \frac{55}{12}(-\frac{2}{5})^n u[n] + \frac{27}{20}(\frac{3}{5})^n u[n] \end{aligned}$$

2.7. We first re-write the system function $H(e^{j\omega})$:

$$\begin{aligned} H(e^{j\omega}) &= e^{j\pi/4} \cdot e^{-j\omega}\left(\frac{1+e^{-j2\omega}+4e^{-j4\omega}}{1+\frac{1}{2}e^{-j2\omega}}\right) \\ &= e^{j\pi/4}G(e^{j\omega}) \end{aligned}$$

Let $y_1[n] = x[n] * g[n]$, then

$$\begin{aligned} x[n] &= \cos(\frac{\pi n}{2}) = \frac{e^{j\pi n/2} + e^{-j\pi n/2}}{2} \\ y_1[n] &= \frac{G(e^{j\pi/2})e^{j\pi n/2} + G(e^{-j\pi/2})e^{-j\pi n/2}}{2} \end{aligned}$$

Evaluating the frequency response at $\omega = \pm\pi/2$:

$$\begin{aligned} G(e^{j\frac{\pi}{2}}) &= e^{-j\frac{\pi}{2}}\left(\frac{1+e^{-j\pi}+4e^{-j2\pi}}{1+\frac{1}{2}e^{-j\pi}}\right) = 8e^{-j\pi/2} \\ G(e^{-j\frac{\pi}{2}}) &= 8e^{j\pi/2} \end{aligned}$$

Therefore,

$$y_1[n] = (8e^{j(\pi n/2-\pi/2)} + 8e^{j(-\pi n/2+\pi/2)})/2 = 8\cos(\frac{\pi}{2}n - \frac{\pi}{2})$$

and

$$y[n] = e^{j\pi/4}y_1[n] = 8e^{j\pi/4}\cos(\frac{\pi}{2}n - \frac{\pi}{2})$$

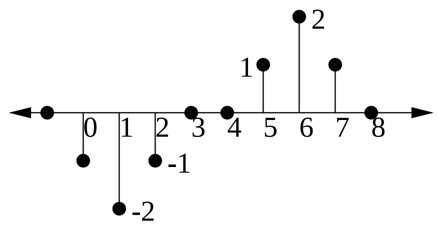
2.8. (a) Notice that

$$x[n] = x_0[n-2] + 2x_0[n-4] + x_0[n-6]$$

Since the system is LTI,

$$y[n] = y_0[n-2] + 2y_0[n-4] + y_0[n-6],$$

and we get sequence shown here:



(b) Since

$$y_0[n] = -1x_0[n+1] + x_0[n-1] = x_0[n] * (-\delta[n+1] + \delta[n-1]),$$
$$h[n] = -\delta[n+1] + \delta[n-1]$$

2.9. For $(-1 < a < 0)$, we have

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

(a) real part of $X(e^{j\omega})$:

$$\begin{aligned} X_R(e^{j\omega}) &= \frac{1}{2} \cdot [X(e^{j\omega}) + X^*(e^{j\omega})] \\ &= \frac{1 - a\cos(\omega)}{1 - 2a\cos(\omega) + a^2} \end{aligned}$$

(b) imaginary part:

$$\begin{aligned} X_I(e^{j\omega}) &= \frac{1}{2j} \cdot [X(e^{j\omega}) - X^*(e^{j\omega})] \\ &= \frac{-a\sin(\omega)}{1 - 2a\cos(\omega) + a^2} \end{aligned}$$

(c) magnitude:

$$\begin{aligned} |X(e^{j\omega})| &= [X(e^{j\omega})X^*(e^{j\omega})]^{\frac{1}{2}} \\ &= \left(\frac{1}{1 - 2acos(\omega) + a^2} \right)^{\frac{1}{2}} \end{aligned}$$

(d) phase:

$$\angle X(e^{j\omega}) = \arctan\left(\frac{-a\sin(\omega)}{1 - a\cos(\omega)} \right)$$

2.10. $x[n]$ can be rewritten as:

$$\begin{aligned} x[n] &= cos(\frac{5\pi n}{2}) \\ &= cos(\frac{\pi n}{2}) \\ &= \frac{e^{j\frac{\pi n}{2}}}{2} + \frac{e^{-j\frac{\pi n}{2}}}{2}. \end{aligned}$$

We now use the fact that complex exponentials are eigenfunctions of LTI systems, we get:

$$\begin{aligned} y[n] &= e^{-j\frac{\pi}{8}}\frac{e^{j\frac{\pi n}{2}}}{2} + e^{j\frac{\pi}{8}}\frac{e^{-j\frac{\pi n}{2}}}{2} \\ &= \frac{e^{j(\frac{\pi n}{2}-\frac{\pi}{8})}}{2} + \frac{e^{-j(\frac{\pi n}{2}-\frac{\pi}{8})}}{2} \\ &= cos(\frac{\pi}{2}(n-\frac{1}{4})). \end{aligned}$$

2.11. First $x[n]$ goes through a lowpass filter with cutoff frequency 0.5π. Since the cosine has a frequency of 0.6π, it will be filtered out. The delayed impulse will be filtered to a delayed sinc and the constant will remain unchanged. We thus get:

$$w[n] = 3\frac{sin(0.5\pi(n-5))}{\pi(n-5)} + 2.$$

$y[n]$ is then given by:

$$y[n] = 3\frac{sin(0.5\pi(n-5))}{\pi(n-5)} - 3\frac{sin(0.5\pi(n-6))}{\pi(n-6)}.$$

2.12. Since system 1 is memoryless, it is time invariant. The input, $x[n]$ is periodic in ω, therefore $w[n]$ will also be periodic in ω. As a consequence, $y[n]$ is periodic in ω and so is A.

Solutions – Chapter 3

The z-Transform

3.1. (a)

$$\begin{aligned} H(z) &= \frac{1-\frac{1}{2}z^{-2}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} \\ &= -4+\frac{5+\frac{7}{2}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} \\ &= -4-\frac{2}{1-\frac{1}{2}z^{-1}}+\frac{7}{1-\frac{1}{4}z^{-1}} \\ h[n] &= -4\delta[n]-2\left(\frac{1}{2}\right)^n u[n]+7\left(\frac{1}{4}\right)^n u[n] \end{aligned}$$

(b)

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=x[n]-\frac{1}{2}x[n-2]$$

3.2.

$$H(z)=\frac{3-7z^{-1}+5z^{-2}}{1-\frac{5}{2}z^{-1}+z^{-2}}=5+\frac{1}{1-2z^{-1}}-\frac{3}{1-\frac{1}{2}z^{-1}}$$

$$h[n] \text{ stable } \Rightarrow h[n]=5\delta[n]-2^n u[-n-1]-3\left(\frac{1}{2}\right)^n u[n]$$

(a)

$$\begin{aligned} y[n] &= h[n]*x[n]=\sum_{k=-\infty}^{n} h[k] \\ &= \begin{cases} -\sum_{k=-\infty}^{n} 2^k=-2^{n+1} & n<0 \\ -\sum_{k-\infty}^{-1} 2^k+5-\sum_{k=0}^{n} 3\left(\frac{1}{2}\right)^k=4-3\frac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}}=-2+3\left(\frac{1}{2}\right)^n & n\geq 0 \end{cases} \\ &= -2u[n]+3\left(\frac{1}{2}\right)^n u[n]-2^{n+1}u[-n-1] \end{aligned}$$

(b)

$$\begin{aligned} Y(z) &= \frac{1}{1-z^{-1}}H(z)=-2\frac{1}{1-z^{-1}}+2\frac{1}{1-2z^{-1}}+3\frac{1}{1-\frac{1}{2}z^{-1}}\ , \qquad \frac{1}{2}<|z|<2 \\ y[n] &= -2u[n]-2(2)^n u[-n-1]+3\left(\frac{1}{2}\right)^n u[n] \end{aligned}$$

3.3.

$$Y(z)=\frac{z^{-1}+z^{-2}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}\cdot\frac{2}{1-z^{-1}} \qquad |z|>1$$

Therefore using a contour C that lies outside of $|z|=1$ we get

$$\begin{aligned} y[1] &= \frac{1}{2\pi j}\oint_C \frac{2(z+1)z^n dz}{(z-\frac{1}{2})(z+\frac{1}{3})(z-1)} \\ &= \frac{2(\frac{1}{2}+1)(\frac{1}{2})}{(\frac{1}{2}+\frac{1}{3})(\frac{1}{2}-1)}+\frac{2(-\frac{1}{3}+1)(-\frac{1}{3})}{(-\frac{1}{3}-\frac{1}{2})(-\frac{1}{3}-1)}+\frac{2(1+1)(1)}{(1-\frac{1}{2})(1+\frac{1}{3})} \\ &= -\frac{18}{5}-\frac{2}{5}+6=2 \end{aligned}$$

3.4. (a)

$$X(z) = \frac{z^{10}}{(z-\frac{1}{2})(z-\frac{3}{2})^{10}(z+\frac{3}{2})^2(z+\frac{5}{2})(z+\frac{7}{2})}$$

Stable $\Rightarrow$ ROC includes $|z| = 1$. Therefore, the ROC is $\frac{1}{2} < |z| < \frac{3}{2}$.

(b) $x[-8] = \Sigma[\text{residues of } X(z)z^{-9} \text{ inside } C]$, where C is contour in ROC (say the unit circle).

$$x[8] = \Sigma\left[\text{residues of } \frac{z}{(z-\frac{1}{2})(z-\frac{3}{2})^{10}(z+\frac{3}{2})^2(z+\frac{5}{2})(z+\frac{7}{2})} \text{ inside unit circle}\right]$$

First order pole at $z = \frac{1}{2}$ is only one inside the unit circle. Therefore

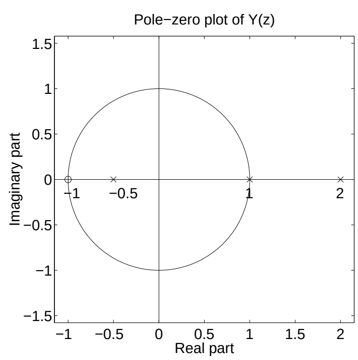
$$x[-8] = \frac{\frac{1}{2}}{(\frac{1}{2}-\frac{3}{2})^{10}(\frac{1}{2}+\frac{3}{2})^2(\frac{1}{2}+\frac{5}{2})(\frac{1}{2}+\frac{7}{2})} = \frac{1}{96}$$

3.5. (a)

$$X(z) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}}$$

The ROC is $\frac{1}{2} < |z| < 2$.

(b) The following figure shows the pole-zero plot of $Y(z)$. Since $X(z)$ has poles at 0.5 and 2, the poles at 1 and -0.5 are due to $H(z)$. Since $H(z)$ is causal, its ROC is $|z| > 1$. The ROC of $Y(z)$ must contain the intersection of the ROC of $X(z)$ and the ROC of $H(z)$. Hence the ROC of $Y(z)$ is $1 < |z| < 2$.



(c)

$$\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{\frac{1+z^{-1}}{(1-z^{-1})(1+\frac{1}{2}z^{-1})(1-2z^{-1})}}{\frac{1}{(1-12z^{-1})(1-2z^{-1})}} \\
&= \frac{(1+z^{-1})(1-\frac{1}{2}z^{-1})}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} \\
&= 1 + \frac{\frac{2}{3}}{1-z^{-1}} + \frac{-\frac{2}{3}}{1+\frac{1}{2}z^{-1}}
\end{aligned}$$

Taking the inverse z-transform, we find

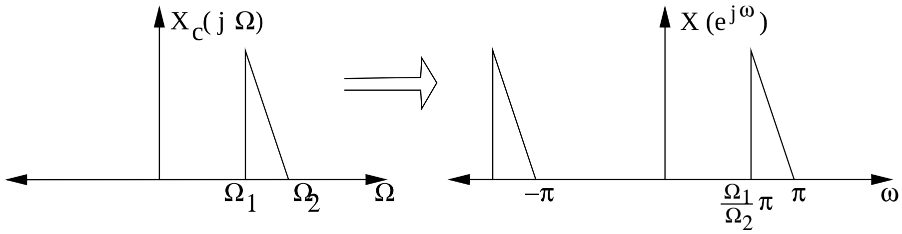
$$h[n] = \delta[n] + \frac{2}{3}u[n] - \frac{2}{3}(-\frac{1}{2})^n u[n]$$

(d) Since $H(z)$ has a pole on the unit circle, the system is not stable.

Solutions – Chapter 4

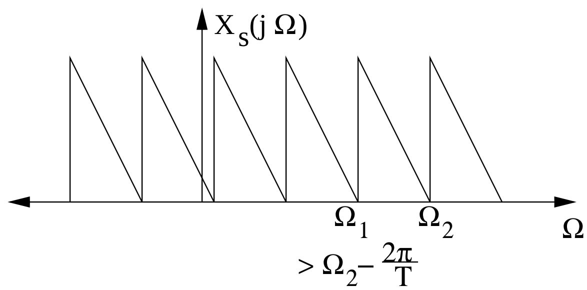
Sampling of Continuous-Time Signals

4.1. (a) Keeping in mind that after sampling, $\omega = \Omega T$, the Fourier transform of $x[n]$ is

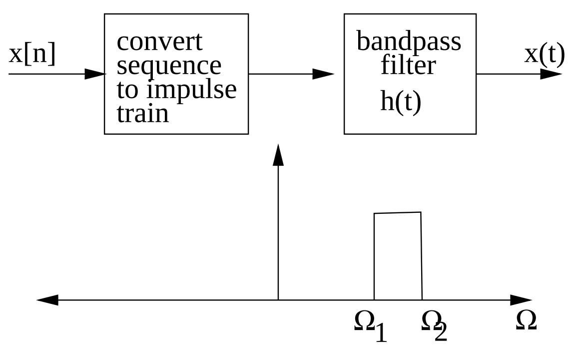


(b) A straight-forward application of the Nyquist criterion would lead to an incorrect conclusion that the sampling rate is at least twice the maximum frequency of $x_c(t)$, or $2\Omega_2$. However, since the spectrum is bandpass, we only need to ensure that the replications in frequency which occur as a result of sampling do not overlap with the original. (See the following figure of $X_s(j\Omega)$.) Therefore, we only need to ensure

$$\Omega_2 - \frac{2\pi}{T} < \Omega_1 \Longrightarrow T < \frac{2\pi}{\Delta\Omega}$$

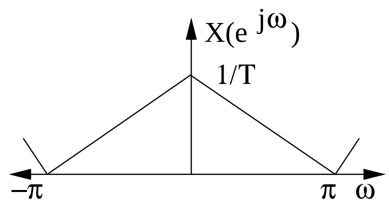


(c) The block diagram along with the frequency response of $h(t)$ is shown here:

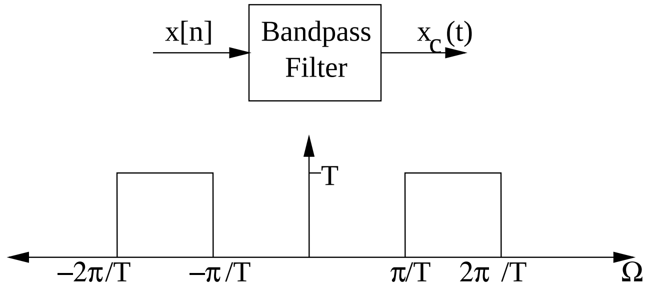


4.2. (a)

$$\omega = \Omega T, \qquad T = \frac{2\pi}{\Omega_0}$$



(b) To recover simply filter out the undesired parts of $X(e^{j\omega})$.



(c)

$$T \le \frac{2\pi}{\Omega_0}$$

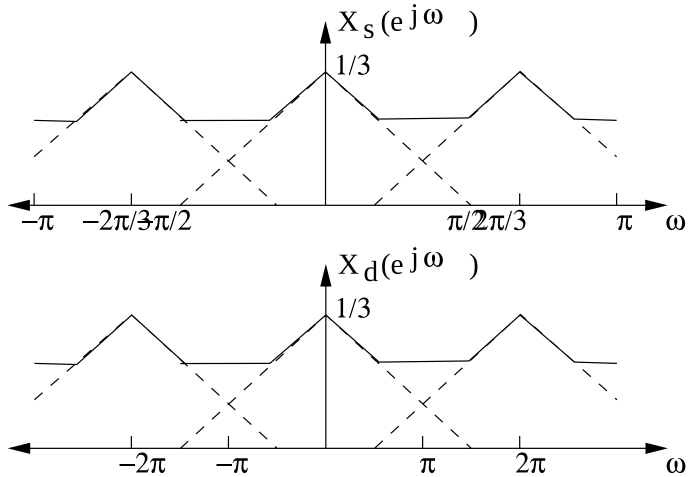
4.3. First we show that $X_s(e^{j\omega})$ is just a sum of shifted versions of $X(e^{j\omega})$:

$$\begin{aligned}
x_s[n] &= \begin{cases} x[n], & n = Mk, \quad k = 0, \pm 1, \pm 2 \\ 0, & \text{otherwise} \end{cases} \\
&= \left(\frac{1}{M} \sum_{k=0}^{M-1} e^{j(2\pi kn/M)} \right) x[n] \\
X_s(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_s[n] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{M} \sum_{k=0}^{M-1} x[n] e^{j(2\pi kn/M)} e^{-j\omega n} \\
&= \frac{1}{M} \sum_{k=0}^{M-1} \sum_{n=-\infty}^{\infty} x[n] e^{-j[\omega - (2\pi k/M)]n} \\
&= \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j[\omega - (2\pi k/M)]} \right)
\end{aligned}$$

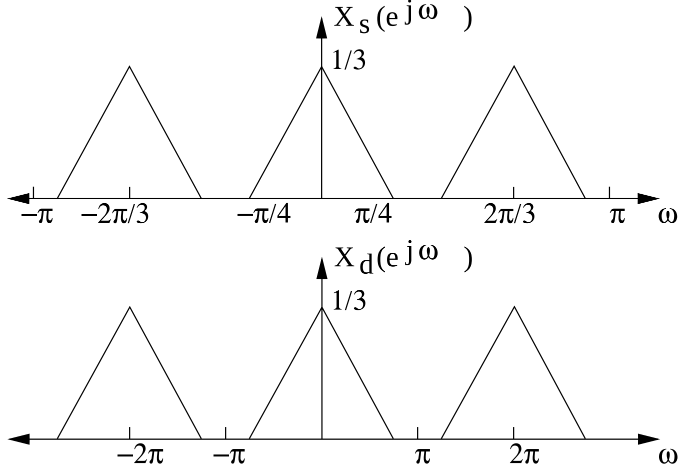
Additionally, $X_d(e^{j\omega})$ is simply $X_s(e^{j\omega})$ with the frequency axis expanded by a factor of M:

$$\begin{aligned}
X_d(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} X_s[Mn] e^{-j\omega n} \\
&= \sum_{l=-\infty}^{\infty} x_s[l] e^{-j(\omega/M)l} \\
&= X_s\left(e^{j(\omega/M)} \right)
\end{aligned}$$

(a) (i) $X_s(e^{j\omega})$ and $X_d(e^{j\omega})$ are sketched below for $M = 3$, $\omega_H = \pi/2$.



(ii) $X_s(e^{j\omega})$ and $X_d(e^{j\omega})$ are sketched below for $M = 3$, $\omega_H = \pi/4$.



(b) From the definition of $X_s(e^{j\omega})$, we see that there will be no aliasing if the signal is bandlimited to π/M. In this problem, $M = 3$. Thus the maximum value of ω_H is $\pi/3$.

4.4. Parseval's Theorem:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

When we upsample, the added samples are zeros, so the upsampled signal $x_u[n]$ has the same energy as the original $x[n]$:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x_u[n]|^2,$$

and by Parseval's theorem:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X_u(e^{j\omega})|^2 d\omega.$$

Hence the amplitude of the Fourier transform does not change.

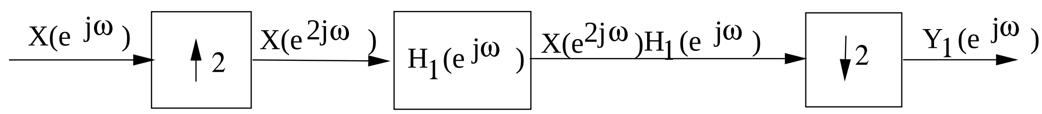
When we downsample, the downsampled signal $x_d[n]$ has less energy than the original $x[n]$ because some samples are discarded. Hence the amplitude of the Fourier transform will change after downsampling.

4.5. (a) Yes, the system is linear because each of the subblocks is linear. The C/D step is defined by $x[n] = x_c(nT)$, which is clearly linear. The DT system is an LTI system. The D/C step consists of converting the sequence to impulses and of CT LTI filtering, both of which are linear.

(b) No, the system is not time-invariant.

For example, suppose that $h[n] = \delta[n]$, $T = 5$ and $x_c(t) = 1$ for $-1 \leq t \leq 1$. Such a system would result in $x[n] = \delta[n]$ and $y_c(t) = \mathrm{sinc}(\pi/5)$. Now suppose we delay the input to be $x_c(t-2)$. Now $x[n] = 0$ and $y_c(t) = 0$.

4.6. We can analyze the system in the frequency domain:



$Y_1(e^{j\omega})$ is $X(e^{2j\omega})H_1(e^{j\omega})$ downsampled by 2:

$$
\begin{aligned}
Y_1(e^{j\omega}) &= \frac{1}{2}\left\{X(e^{2j\omega/2})H_1(e^{j\omega/2}) + X(e^{(2j(\omega-2\pi)/2})H_1(e^{j(\omega-2\pi)/2})\right\} \\
&= \frac{1}{2}\left\{X(e^{j\omega})H_1(e^{j\omega/2}) + X(e^{j(\omega-2\pi)})H_1(e^{j(\frac{\omega}{2}-\pi)})\right\} \\
&= \frac{1}{2}\left\{H_1(e^{j\omega/2}) + H_1(e^{j(\frac{\omega}{2}-\pi)})\right\} X(e^{j\omega}) \\
&= H_2(e^{j\omega})X(e^{j\omega}) \\
H_2(e^{j\omega}) &= \frac{1}{2}\left\{H_1(e^{j\omega/2}) + H_1(e^{j(\frac{\omega}{2}-\pi)})\right\}
\end{aligned}
$$

4.7.

$$
\begin{array}{ll}
X_c(j\Omega) = 0 & |\Omega| \geq 4000\pi \\
Y(j\Omega) = |\Omega| X_c(j\Omega), & 1000\pi \leq |\Omega| \leq 2000\pi
\end{array}
$$

Since only half the frequency band of $X_c(j\Omega)$ is needed, we can alias everything past $\Omega = 2000\pi$. Hence, $T = 1/3000$ s.

Now that T is set, figure out $H(e^{j\omega})$ band edges.

$$
\begin{array}{lll}
\omega_1 = \Omega_1 T & \Rightarrow \omega_1 = 2\pi \cdot 500 \cdot \frac{1}{3000} & \Rightarrow \omega_1 = \dfrac{\pi}{3} \\
\omega_2 = \Omega_2 T & \Rightarrow \omega_2 = 2\pi \cdot 1000 \cdot \frac{1}{3000} & \Rightarrow \omega_2 = \dfrac{2\pi}{3}
\end{array}
$$

$$
H(e^{j\omega}) = \begin{cases} |\omega| & \frac{\pi}{3} \leq |\omega| \leq \frac{2\pi}{3} \\ 0 & 0 \leq |\omega| < \frac{\pi}{3}, \frac{2\pi}{3} < |\omega| \leq \pi \end{cases}
$$

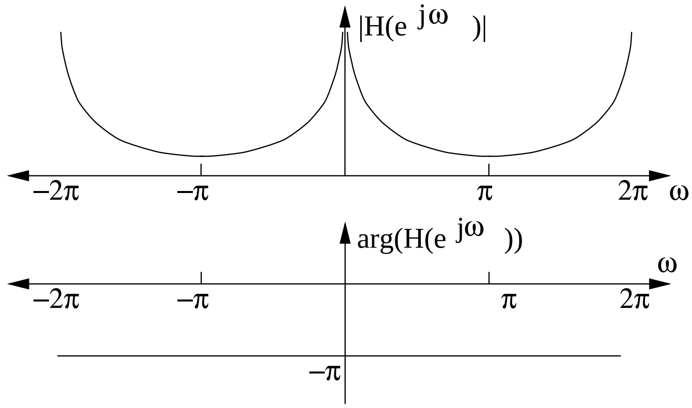
4.8.

$$
X_c(j\Omega) = 0, \qquad |\Omega| > \frac{\pi}{T}
$$

$$
y_r(t) = \int_{-\infty}^{t} x_c(\tau)d\tau \Longrightarrow H_c(j\Omega) = \frac{1}{j\Omega}
$$

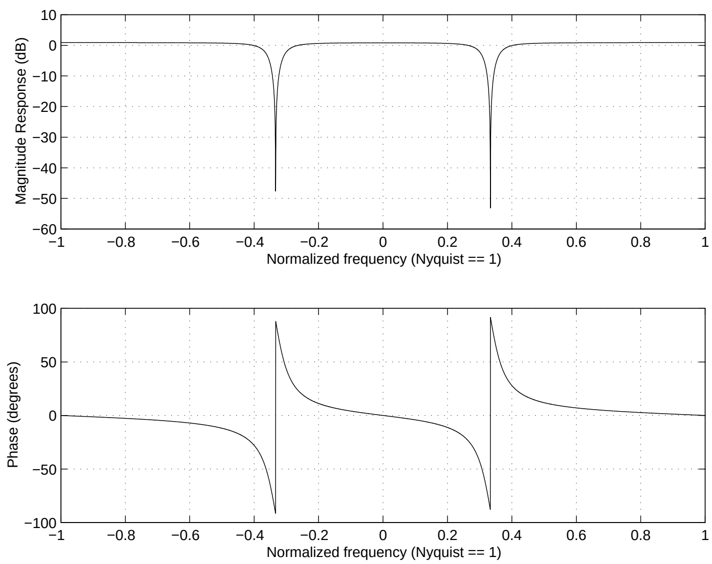
In discrete-time, we want

$$
H(e^{j\omega}) = \begin{cases} \frac{1}{j\omega}, & -\pi \leq \omega \leq \pi \\ 0, & \text{otherwise} \end{cases}
$$



4.9. (a) The highest frequency is $\pi/T = \pi \times 10000$.

(b)



(c) To filter the 60Hz out,

$$\omega_0 = T\Omega = \frac{1}{10,000} \cdot 2\pi \cdot 60 = \frac{3\pi}{250}$$

4.10. (a) Since there is no aliasing involved in this process, we may choose T to be any value. Choose $T = 1$ for simplicity. $X_c(j\Omega) = 0, |\Omega| \geq \pi/T$. Since $Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$, $Y_c(j\Omega) = 0, |\Omega| \geq \pi/T$. Therefore, there will be no aliasing problems in going from $y_c(t)$ to $y[n]$.

Recall the relationship $\omega = \Omega T$. We can simply use this in our system conversion:

$$\begin{aligned} H(e^{j\omega}) &= e^{-j\omega/2} \\ H(j\Omega) &= e^{-j\Omega T/2} \\ &= e^{-j\Omega/2}, \qquad T = 1 \end{aligned}$$

Note that the choice of T and therefore $H(j\Omega)$ is not unique.

(b)

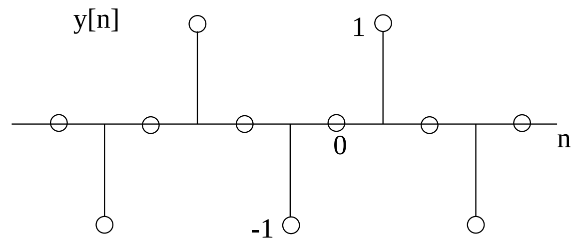
$$\begin{aligned} \cos\left(\frac{5\pi}{2}n - \frac{\pi}{4}\right) &= \frac{1}{2}\left[e^{j(\frac{5\pi}{2}n - \frac{\pi}{4})} + e^{-j(\frac{5\pi}{2}n - \frac{\pi}{4})}\right] \\ &= \frac{1}{2}e^{-j(\pi/4)}e^{j(5\pi/2)n} + \frac{1}{2}e^{j(\pi/4)}e^{-j(5\pi/2)n} \end{aligned}$$

Since $H(e^{j\omega})$ is an LTI system, we can find the response to each of the two eigenfunctions separately.

$$y[n] = \frac{1}{2}e^{-j(\pi/4)}H\left(e^{j(5\pi/2)}\right)e^{j(5\pi/2)n} + \frac{1}{2}e^{j(\pi/4)}H\left(e^{-j(5\pi/2)}\right)e^{-j(5\pi/2)n}$$

Since $H(e^{j\omega})$ is defined for $0 \le |\omega| \le \pi$ we must evaluate the frequency at the baseband, i.e., $5\pi/2 \Rightarrow 5\pi/2 - 2\pi = \pi/2$. Therefore,

$$\begin{aligned} y[n] &= \frac{1}{2}e^{-j(\pi/4)}H\left(e^{j(5\pi/2)}\right)e^{j(5\pi/2)n} + \frac{1}{2}e^{j(\pi/4)}H\left(e^{-j(5\pi/2)}\right)e^{-j(5\pi/2)n} \\ &= \frac{1}{2}\left(e^{j[(5\pi/2)n-(\pi/2)]} + e^{-j[(5\pi/2)n-(\pi/2)]}\right) \\ &= \cos\left(\frac{5\pi}{2}n - \frac{\pi}{2}\right) \end{aligned}$$



4.11. The frequency response $H(e^{j\omega}) = H_c(j\Omega/T)$. Finding that

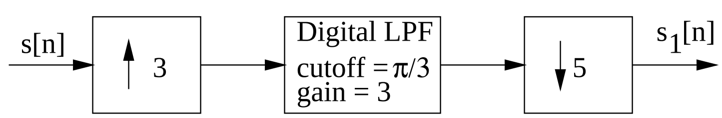
$$H_c(j\Omega) = \frac{1}{(j\Omega)^2 + 4(j\Omega) + 3},$$

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{(10j\omega)^2 + 4(10j\omega) + 3} \\ &= \frac{1}{-100\omega^2 + 3 + 40j\omega} \end{aligned}$$

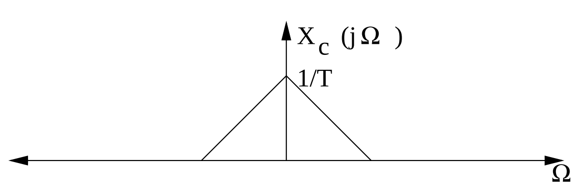
4.12. (a) Since $\Omega T = \omega$, $\quad (2\pi \cdot 100)T = \frac{\pi}{2} \Rightarrow T = \frac{1}{400}$

(b) The downsampler has $M = 2$. Since $x[n]$ is bandlimited to $\frac{\pi}{M}$, there will be no aliasing. The frequency axis simply expands by a factor of 2.
For $y_c(t) = x_c(t) \Leftrightarrow Y_c(j\Omega) = X_c(j\Omega)$.
Therefore $\Omega T' \Rightarrow 2\pi \cdot 100T' \Rightarrow T' = \frac{1}{200}$.

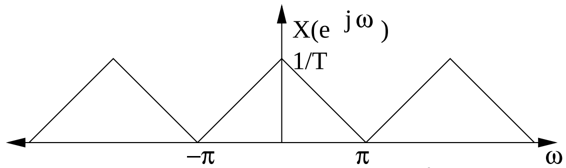
4.13. In both systems, the speech was filtered first so that the subsequent sampling results in no aliasing. Therefore, going $s[n]$ to $s_1[n]$ basically requires changing the sampling rate by a factor of 3kHz/5kHz = 3/5. This is done with the following system:



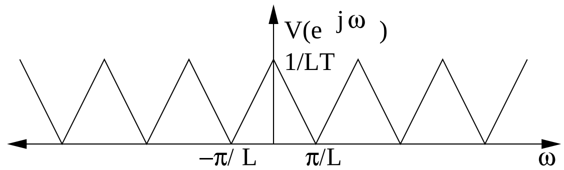
4.14. $X_c(j\Omega)$ is drawn below.



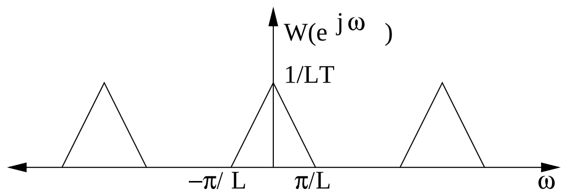
$x_c(t)$ is sampled at sampling period T, so there is no aliasing in $x[n]$.



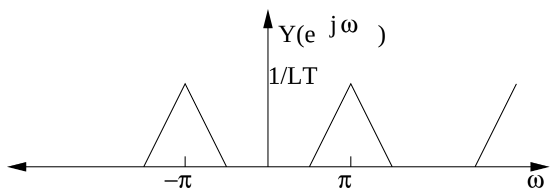
Inserting $L-1$ zeros between samples compresses the frequency axis.



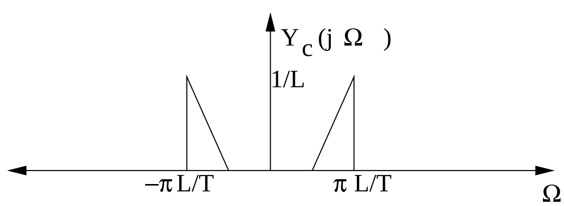
The filter $H(e^{j\omega})$ removes frequency components between π/L and π.



The multiplication by $(-1)^n$ shifts the center of the frequency band from 0 to π.



The D/C conversion maps the range $-\pi$ to π to the range $-\pi/T$ to π/T.



4.15. (a)

$$h[n] = 0, \qquad |n| > (RL-1)$$

Therefore, for causal system delay by $RL-1$ samples.

(b) General interpolator condition:

$$\begin{aligned} h[0] &= 1 \\ h[kL] &= 0, \qquad k = \pm 1, \pm 2, \ldots \end{aligned}$$

(c)

$$y[n] = \sum_{k=-(RL-1)}^{(RL-1)} h[k]v[n-k] = h[0]v[n] + \sum_{k=1}^{RL-1} h[n](v[n-k] + v[n+k])$$

This requires only RL-1 multiplies, (assuming $h[0] = 1$.)

(d)

$$y[n] = \sum_{k=n-(RL-1)}^{n+(RL-1)} v[k]h[n-k]$$

If $n = mL$ (m an integer), then we don't have any multiplications since $h[0] = 1$ and the other non-zero samples of $v[k]$ hit at the zeros $h[n]$. Otherwise the impulse response spans $2RL - 1$ samples of $v[n]$, but only $2R$ of these are non-zero. Therefore, there are $2R$ multiplies.

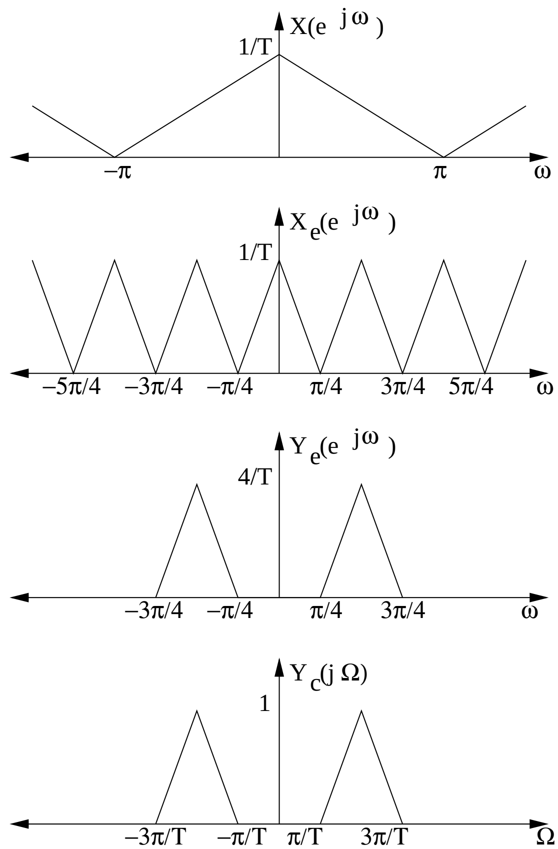
4.16. (a) See figures below.

(b) From part(a), we see that

$$Y_c(j\Omega) = X_c(j(\Omega - \frac{2\pi}{T})) + X_c(j(\Omega + \frac{2\pi}{T}))$$

Therefore,

$$y_c(t) = 2x_c(t)\cos(\frac{2\pi}{T}t)$$



4.17. (a) The Nyquist criterion states that $x_c(t)$ can be recovered as long as

$$\frac{2\pi}{T} \geq 2 \times 2\pi(250) \Longrightarrow T \leq \frac{1}{500}.$$

In this case, $T = 1/500$, so the Nyquist criterion is satisfied, and $x_c(t)$ can be recovered.

(b) Yes. A delay in time does not change the bandwidth of the signal. Hence, $y_c(t)$ has the same bandwidth and same Nyquist sampling rate as $x_c(t)$.

(c) Consider first the following expressions for $X(e^{j\omega})$ and $Y(e^{j\omega})$:

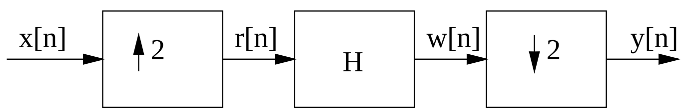
$$\begin{aligned}
X(e^{j\omega}) &= \frac{1}{T}X_c(j\Omega)\mid_{\Omega=\frac{\omega}{T}} = \frac{1}{500}X_c(j500\omega) \\
Y(e^{j\omega}) &= \frac{1}{T}Y_c(j\Omega)\mid_{\Omega=\frac{\omega}{T}} = \frac{1}{T}e^{-j\Omega/1000}X_c(j\Omega)\mid_{\Omega=\frac{\omega}{T}} \\
&= \frac{1}{500}e^{-j\omega/2}X_c(j500\omega) \\
&= e^{-j\omega/2}X(e^{j\omega})
\end{aligned}$$

Hence, we let

$$H(e^{j\omega}) = \begin{cases} 2e^{-j\omega}, & |\omega| < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

Then, in the following figure,

$$\begin{aligned}
R(e^{j\omega}) &= X(e^{j2\omega}) \\
W(e^{j\omega}) &= \begin{cases} 2e^{-j\omega}X(e^{j2\omega}), & |\omega| < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases} \\
Y(e^{j\omega}) &= e^{-j\omega/2}X(e^{j\omega})
\end{aligned}$$



(d) Yes, from our analysis above,

$$H_2(e^{j\omega}) = e^{-j\omega/2}$$

4.18. (a) Notice first that

$$X_c(j\Omega) = \begin{cases} F_c(j\Omega)|H_{aa}(j\Omega)|e^{-j\Omega^3}, & |\Omega| \le 400\pi \\ E_c(j\Omega)|H_{aa}(j\Omega)|e^{-j\Omega^3}, & 400\pi \le |\Omega| \le 800\pi \\ 0, & \text{otherwise} \end{cases}$$

For the given $T = 1/800$, there is no aliasing from the C/D conversion. Hence, the equivalent CT transfer function $H_c(j\Omega)$ can be written as

$$H_c(j\Omega) = \begin{cases} H(e^{j\omega})|_{\omega=\Omega T}, & |\Omega| \le \pi/T \\ 0, & \text{otherwise} \end{cases}$$

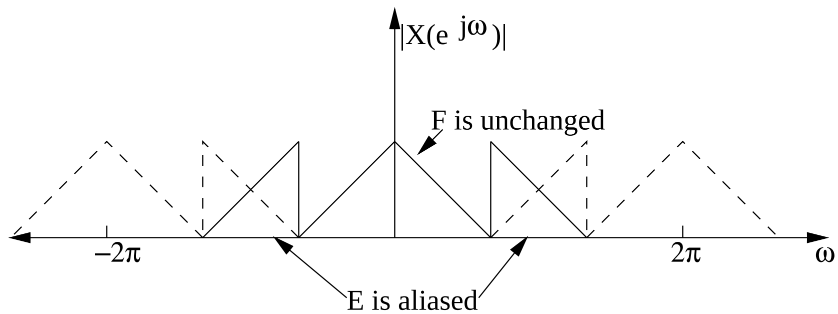
Furthermore, since $Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$, the desired tranfer function is

$$H_c(j\Omega) = \begin{cases} e^{j\Omega^3}, & |\Omega| \le 400\pi \\ 0, & \text{otherwise} \end{cases}$$

Combining the two previous equations, we find

$$H(e^{j\omega}) = \begin{cases} e^{j(800\omega)^3}, & |\omega| \le \pi/2 \\ 0, & \pi/2 \le |\omega| \le \pi \end{cases}$$

(b) Some aliasing will occur if $2\pi/T < 1600\pi$. However, this is fine as long as the aliasing affects only $E_c(j\Omega)$ and not $F_c(j\Omega)$, as we show below:



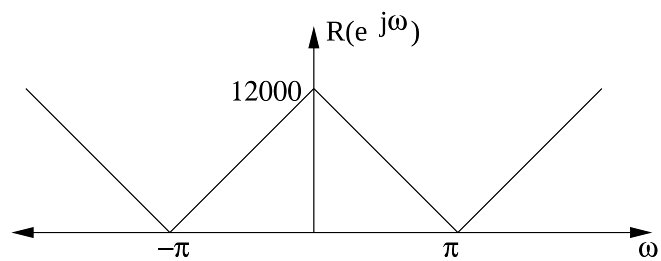
In order for the aliasing to not affect $F_c(j\Omega)$, we require

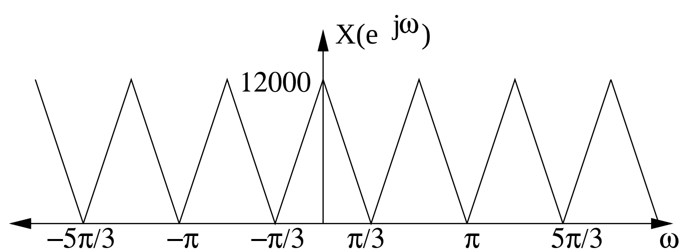
$$\frac{2\pi}{T} - 800\pi \geq 400\pi \Longrightarrow \frac{2\pi}{T} \geq 1200\pi$$

The minimum $\frac{2\pi}{T}$ is 1200π. For this choice, we get

$$H(e^{j\omega}) = \begin{cases} e^{j(600\omega)^3}, & |\omega| \leq 2\pi/3 \\ 0, & 2\pi/3 \leq |\omega| \leq \pi \end{cases}$$

4.19. (a) See the following figure:

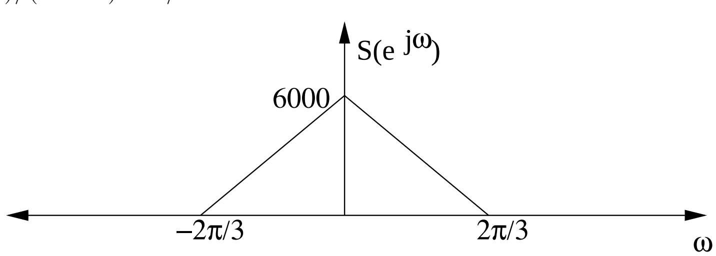




(b) For this to be true, $H(e^{j\omega})$ needs to filter out $X(e^{j\omega})$ for $\pi/3 \leq |\omega| \leq \pi$. Hence let $\omega_0 = \pi/3$. Furthermore, we want

$$\frac{\pi/2}{T_2} = 2\pi(1000) \Longrightarrow T_2 = 1/6000$$

(c) Matching the following figure of $S(e^{j\omega})$ with the figure for $R_c(j\Omega)$, and remembering that $\Omega = \omega/T$, we get $T_3 = (2\pi/3)/(2000\pi) = 1/3000$.



4.20. Notice first that since $x_c(t)$ is time-limited,

$$A = \int_0^{10} x_c(t)dt = \int_{-\infty}^{\infty} x_c(t)dt = X_c(j\Omega)|_{\Omega=0}.$$

To estimate $X_c(j \cdot 0)$ by DT processing, we need to sample only fast enough so that $X_c(j \cdot 0)$ is not aliased. Hence, we pick

$$2\pi/T = 2\pi \times 10^4 \Longrightarrow T = 10^{-4}.$$

The resulting spectrum satisfies

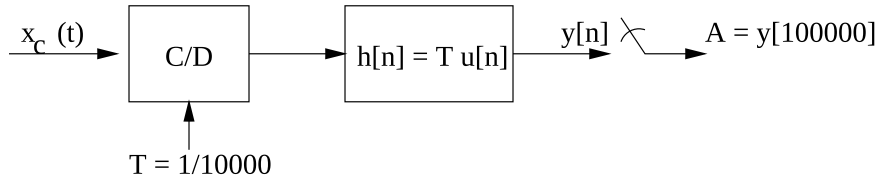
$$X(e^{j \cdot 0}) = \frac{1}{T} X_c(j \cdot 0)$$

Further,

$$X(e^{j \cdot 0}) = \sum_{n=-\infty}^{\infty} x[n].$$

Therefore, we pick $h[n] = Tu[n]$, which makes the system an accumulator. Our estimate $\hat{A}$ is the output $y[n]$ at $n = 10/(10^{-4}) = 10^5$, when all of the non-zero samples of $x[n]$ have been added-up. This is an *exact* estimate given our assumption of both band- and time-limitedness. Since the assumption can never be exactly satisfied, however, this method only gives an approximate estimate for actual signals.

The overall system is as follows:



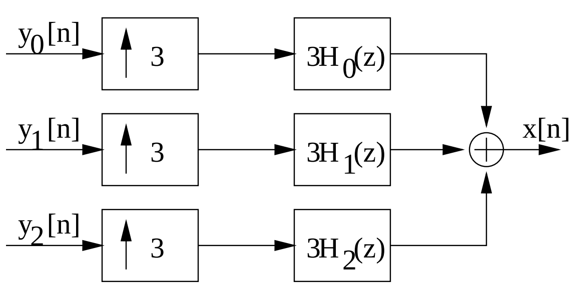
4.21. (a) Notice that

$$\begin{aligned} y_0[n] &= x[3n] \\ y_1[n] &= x[3n+1] \\ y_2[n] &= x[3n+2], \end{aligned}$$

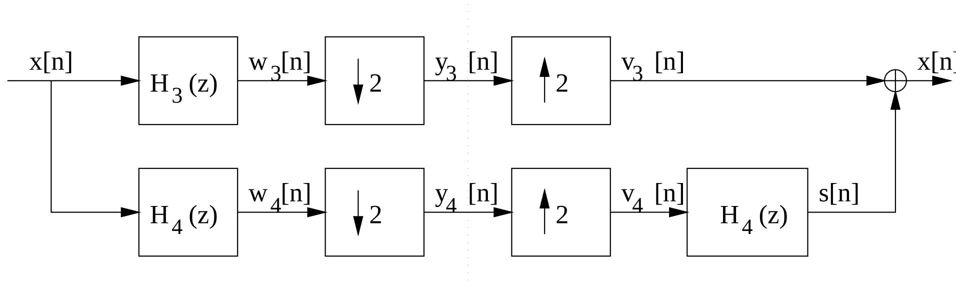
and therefore,

$$x[n] = \begin{cases} y_0[n/3], & n = 3k \\ y_1[(n-1)/3], & n = 3k+1 \\ y_2[(n-2)/3], & n = 3k+2 \end{cases}$$

(b) Yes. Since the bandwidth of the filters are $2\pi/3$, there is no aliasing introduced by downsampling. Hence to reconstruct $x[n]$, we need the system shown in the following figure:



(c) Yes, $x[n]$ can be reconstructed from $y_3[n]$ and $y_4[n]$ as demonstrated by the following figure:



In the following discussion, let $x_e[n]$ denote the even samples of $x[n]$, and $x_o[n]$ denote the odd samples of $x[n]$:

$$x_e[n] = \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$x_o[n] = \begin{cases} 0, & n \text{ even} \\ x[n], & n \text{ odd} \end{cases}$$

In the figure, $y_3[n] = x[2n]$, and hence,

$$\begin{aligned} v_3[n] &= \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\ &= x_e[n] \end{aligned}$$

Furthermore, it can be verified using the IDFT that the impulse response $h_4[n]$ corresponding to $H_4(e^{j\omega})$ is

$$h_4[n] = \begin{cases} -2/(j\pi n), & n \text{ odd} \\ 0, & \text{otherwise} \end{cases}$$

Notice in particular that every other sample of the impulse response $h_4[n]$ is zero. Also, from the form of $H_4(e^{j\omega})$, it is clear that $H_4(e^{j\omega})H_4(e^{j\omega}) = 1$, and hence $h_4[n] * h_4[n] = \delta[n]$.
Therefore,

$$\begin{aligned} v_4[n] &= \begin{cases} y_4[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\ &= \begin{cases} w_4[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\ &= \begin{cases} (x * h_4)[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\ &= x_o[n] * h_4[n] \end{aligned}$$

where the last equality follows from the fact that $h_4[n]$ is non-zero only in the odd samples.
Now, $s[n] = v_4[n] * h_4[n] = x_o[n] * h_4[n] * h_4[n] = x_o[n]$, and since $x[n] = x_e[n] + x_o[n]$, $s[n] + v_3[n] = x[n]$.

Solutions – Chapter 5

Transform Analysis of Linear Time-Invariant Systems

5.1.

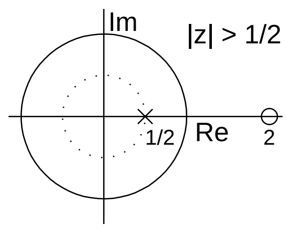
$$H(z) = \frac{1 - a^{-1}z^{-1}}{1 - az^{-1}} = \frac{Y(z)}{X(z)}, \qquad \text{causal, so ROC is } |z| > a$$

(a) Cross multiplying and taking the inverse transform

$$y[n] - ay[n-1] = x[n] - \frac{1}{a}x[n-1]$$

(b) Since $H(z)$ is causal, we know that the ROC is $|z| > a$. For stability, the ROC must include the unit circle. So, $H(z)$ is stable for $|a| < 1$.

(c) $a = \frac{1}{2}$



(d)

$$H(z) = \frac{1}{1 - az^{-1}} - \frac{a^{-1}z^{-1}}{1 - az^{-1}}, \qquad |z| > a$$

$$h[n] = (a)^n u[n] - \frac{1}{a}(a)^{n-1}u[n-1]$$

(e)

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \frac{1 - a^{-1}e^{-j\omega}}{1 - ae^{-j\omega}}$$

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = \frac{1 - a^{-1}e^{-j\omega}}{1 - ae^{-j\omega}} \cdot \frac{1 - a^{-1}e^{j\omega}}{1 - ae^{j\omega}}$$

$$\begin{aligned} |H(e^{j\omega})| &= \left(\frac{1 + \frac{1}{a^2} - \frac{2}{a}\cos\omega}{1 + a^2 - 2a\cos\omega} \right)^{\frac{1}{2}} \\ &= \frac{1}{a}\left(\frac{a^2 + 1 - 2a\cos\omega}{1 + a^2 - 2a\cos\omega} \right)^{\frac{1}{2}} \\ &= \frac{1}{a} \end{aligned}$$

5.2. (a) Type I:

$$A(\omega) = \sum_{n=0}^{M/2} a[n]\cos\omega n$$

$\cos 0 = 1,\ \cos\pi = -1$, so there are no restrictions.

Type II:

$$A(\omega) = \sum_{n=1}^{(M+1)/2} b[n] \cos \omega \left(n - \frac{1}{2} \right)$$

$\cos 0 = 1,\ \cos \left(n\pi - \frac{\pi}{2} \right) = 0.$ So $H(e^{j\pi}) = 0.$

Type III:

$$A(\omega) = \sum_{n=0}^{M/2} c[n] \sin \omega n$$

$\sin 0 = 0,\ \sin n\pi = 0$, so $H(e^{j0}) = H(e^{j\pi}) = 0.$

Type IV:

$$A(\omega) = \sum_{n=1}^{(M+1)/2} d[n] \sin \omega \left(n - \frac{1}{2} \right)$$

$\sin 0 = 0,\ \sin \left(n\pi - \frac{\pi}{2} \right) \neq 0$, so just $H(e^{j0}) = 0.$

(b)

	Type I	Type II	Type III	Type IV
Lowpass	Y	Y	N	N
Bandpass	Y	Y	Y	Y
Highpass	Y	N	N	Y
Bandstop	Y	N	N	N
Differentiator	Y	N	N	Y

5.3. (a) Taking the z-transform of both sides and rearranging

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{4} + z^{-2}}{1 - \frac{1}{4} z^{-2}}$$

Since the poles and zeros {2 poles at $z = \pm 1/2$, 2 zeros at $z = \pm 2$} occur in conjugate reciprocal pairs the system is allpass. This property is easy to recognize since, as in the system above, the coefficients of the numerator and denominator z-polynomials get reversed (and in general conjugated).

(b) It is a property of allpass systems that the output energy is equal to the input energy. Here is the proof.

$$
\begin{aligned}
\sum_{n=0}^{N-1} |y[n]|^2 &= \sum_{n=-\infty}^{\infty} |y[n]|^2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|Y(e^{j\omega})\right|^2 d\omega && \text{(by Parseval's Theorem)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H(e^{j\omega})X(e^{j\omega})\right|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega && (\left|H(e^{j\omega})\right|^2 = 1 \text{ since } h[n] \text{ is allpass}) \\
&= \sum_{n=-\infty}^{\infty} |x[n]|^2 && \text{(by Parseval's theorem)} \\
&= \sum_{n=0}^{N-1} |x[n]|^2 \\
&= 5
\end{aligned}
$$

5.4. *The statement is false.* A non-causal system can indeed have a positive constant group delay. For example, consider the non-causal system

$$h[n] = \delta[n+1] + \delta[n] + 4\delta[n-1] + \delta[n-2] + \delta[n-3]$$

This system has the frequency response

$$
\begin{aligned}
H(e^{j\omega}) &= e^{j\omega} + 1 + 4e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} \\
&= e^{-j\omega}(e^{j2\omega} + e^{j\omega} + 4 + e^{-j\omega} + e^{-j2\omega}) \\
&= e^{-j\omega}(4 + 2\cos(\omega) + 2\cos(2\omega)) \\
\left|H(e^{j\omega})\right| &= 4 + 2\cos(\omega) + 2\cos(2\omega) \\
\angle H(e^{j\omega}) &= -\omega \\
\text{grd}[H(e^{j\omega})] &= 1
\end{aligned}
$$

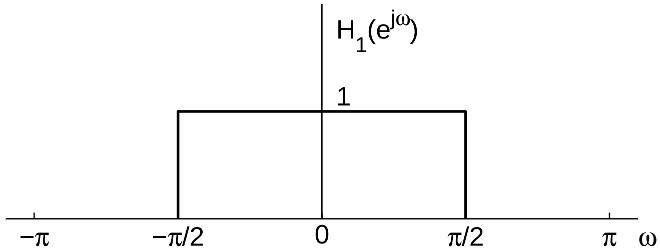
5.5. Making use of some DTFT properties can aide in the solution of this problem. First, note that

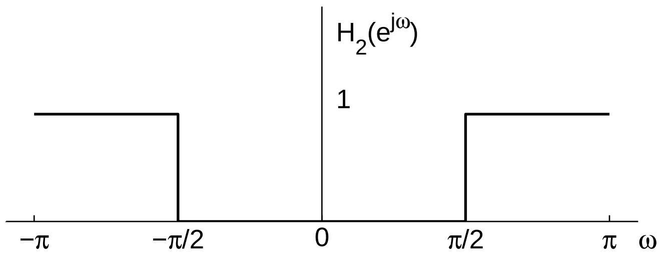
$$
\begin{aligned}
h_2[n] &= (-1)^n h_1[n] \\
h_2[n] &= e^{-j\pi n} h_1[n]
\end{aligned}
$$

Using the DTFT property that states that modulation in the time domain corresponds to a shift in the frequency domain,

$$H_2(e^{j\omega}) = H_1(e^{j(\omega+\pi)})$$

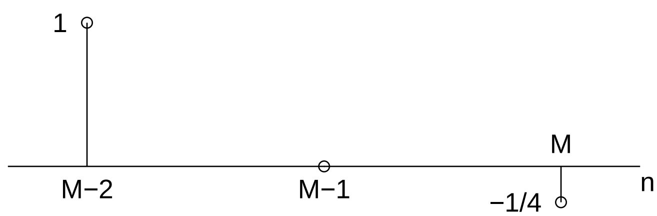
Consequently, $H_2(e^{j\omega})$ is simply $H_1(e^{j\omega})$ shifted by π. The ideal low pass filter has now become the ideal high pass filter, as shown below.





5.6. (a)

$$H(z) = \frac{(z+\frac{1}{2})(z-\frac{1}{2})}{z^M} = z^{-(M-2)}\left(1-\frac{1}{4}z^{-2}\right)$$



(b)

$$\begin{aligned} w[n] &= x[n-(M-2)] - \frac{1}{4}x[n-M] \\ y[n] &= w[2n] = x[2n-(M-2)] - \frac{1}{4}x[2n-M] \end{aligned}$$

Let $v[n] = x[2n]$,

$$y[n] = v[n-(M-2)/2] - \frac{1}{4}v[n-(M/2)]$$

Therefore,

$$g[n] = \delta[n-(M-2)/2] - \frac{1}{4}\delta[n-(M/2)], \qquad M \text{ even}$$

$$G(z) = z^{-(M-2)/2} - \frac{1}{4}z^{-M/2}$$

5.7. (a)

$$H(z) = \frac{z^{-2}}{(1-\frac{1}{2}z^{-1})(1-3z^{-1})}, \qquad \text{stable, so the ROC is } \tfrac{1}{2} < |z| < 3$$

$$x[n] = u[n] \Leftrightarrow X(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1$$

$$Y(z) = X(z)H(z) = \frac{\frac{4}{5}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{5}}{1 - 3z^{-1}} - \frac{1}{1 - z^{-1}}, \qquad 1 < |z| < 3$$

$$y[n] = \frac{4}{5}\left(\frac{1}{2}\right)^n u[n] - \frac{1}{5}(3)^n u[-n-1] - u[n]$$

(b) ROC includes $z = \infty$ so $h[n]$ is causal. Since both $h[n]$ and $x[n]$ are 0 for $n < 0$, we know that $y[n]$ is also 0 for $n < 0$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2}}{1 - \frac{7}{2}z^{-1} + \frac{3}{2}z^{-2}}$$

$$Y(z) - \frac{7}{2}z^{-1}Y(z) + \frac{3}{2}z^{-2}Y(z) = z^{-2}X(z)$$

$$y[n] = x[n-2] + \frac{7}{2}y[n-1] - \frac{3}{2}y[n-2]$$

Since $y[n] = 0$ for $n < 0$, recursion can be done:

$$y[0] = 0, \qquad y[1] = 0, \qquad y[2] = 1$$

(c)

$$H_i(z) = \frac{1}{H(z)} = z^2 - \frac{7}{2}z + \frac{3}{2}, \qquad \text{ROC: entire z-plane}$$

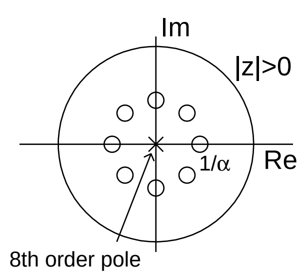
$$h_i[n] = \delta[n+2] - \frac{7}{2}\delta[n+1] + \frac{3}{2}\delta[n]$$

5.8. (a)

$$X(z) = S(z)(1 - e^{-8\alpha}z^{-8})$$

$$H_1(z) = 1 - e^{-8\alpha}z^{-8}$$

There are 8 zeros at $z = e^{-\alpha}e^{j\frac{\pi}{4}k}$ for $k = 0, \ldots, 7$
and 8 poles at the origin.



(b)

$$Y(z) = H_2(z)X(z) = H_2(z)H_1(z)S(z)$$

$$H_2(z) = \frac{1}{H_1(z)} = \frac{1}{1 - e^{-8\alpha}z^{-8}}$$

$$|z| > e^{-\alpha} \quad \text{stable and causal}, \qquad |z| < e^{-\alpha} \quad \text{not causal or stable}$$

(c) Only the causal $h_2[n]$ is stable, therefore only it can be used to recover $s[n]$.

$$h[n] = \begin{cases} e^{-\alpha n}, & n = 0, 8, 16, \ldots \\ 0, & \text{otherwise} \end{cases}$$

(d)

$$s[n] = \delta[n] \Rightarrow x[n] = \delta[n] - e^{-8\alpha}\delta[n-8]$$

$$\begin{aligned} x[n] * h_2[n] &= \delta[n] - e^{-8\alpha}\delta[n-8] \\ &\quad + e^{-8\alpha}(\delta[n-8] - e^{-8\alpha}\delta[n-16]) \\ &\quad + e^{-16\alpha}(\delta[n-16] - e^{-8\alpha}\delta[n-32]) + \cdots \\ &= \delta[n] \end{aligned}$$

5.9.

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$$

(a)

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - \frac{1}{3}z^{-1}} = \frac{2 - \frac{5}{6}z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}, \quad |z| > \frac{1}{2}$$

Since $h[n], x[n] = 0$ for $n < 0$ we can assume initial rest conditions.

$$y[n] = \frac{5}{6}y[n-1] - \frac{1}{6}y[n-2] + 2x[n] - \frac{5}{6}x[n-1]$$

(b)

$$h_1[n] = \begin{cases} h[n], & n \le 10^9 \\ 0, & n > 10^9 \end{cases}$$

(c)

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{m=0}^{N-1} h[m]z^{-m}, \quad N = 10^9 + 1$$

$$y[n] = \sum_{m=0}^{N-1} h[m]x[n-m]$$

(d) For IIR, we have 4 multiplies and 3 adds per output point. This gives us a total of $4N$ multiplies and $3N$ adds. So, IIR grows with order N. For FIR, we have N multiplies and $N-1$ adds for the n^{th} output point, so this configuration has order N^2.

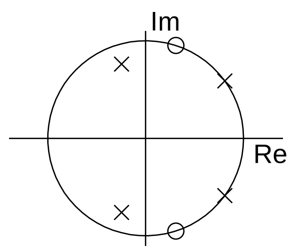
5.10. (a)

$$20\log_{10}|H(e^{j(\pi/5)})| = \infty \Rightarrow \text{pole at } e^{j(\pi/5)}$$

$$20\log_{10}|H(e^{j(2\pi/5)})| = -\infty \Rightarrow \text{zero at } e^{j(2\pi/5)}$$

Resonance at $\omega = \frac{3\pi}{5} \Rightarrow$ pole inside unit circle here.

Since the impulse response is real, the poles and zeros must be in conjugate pairs. The remaining 2 zeros are at zero (the number of poles always equals the number of zeros).



(b) Since $H(z)$ has poles, we know $h[n]$ is IIR.

(c) Since $h[n]$ is causal and IIR, it cannot be symmetric, and thus cannot have linear phase.

(d) Since there is a pole at $|z| = 1$, the ROC does not include the unit circle. This means the system is not stable.

5.11. Convolving two symmetric sequences yields another symmetric sequence. A symmetric sequence convolved with an antisymmetric sequence gives an antisymmetric sequence. If you convolve two antisymmetric sequences, you will get a symmetric sequence.

$$A: \; h_1[n] * h_2[n] * h_3[n] = (h_1[n] * h_2[n]) * h_3[n]$$

$h_1[n] * h_2[n]$ is symmetric about $n = 3$, $\quad (-1 \le n \le 7)$

$(h_1[n] * h_2[n]) * h_3[n]$ is antisymmetric about $n = 3$, $\quad (-3 \le n \le 9)$

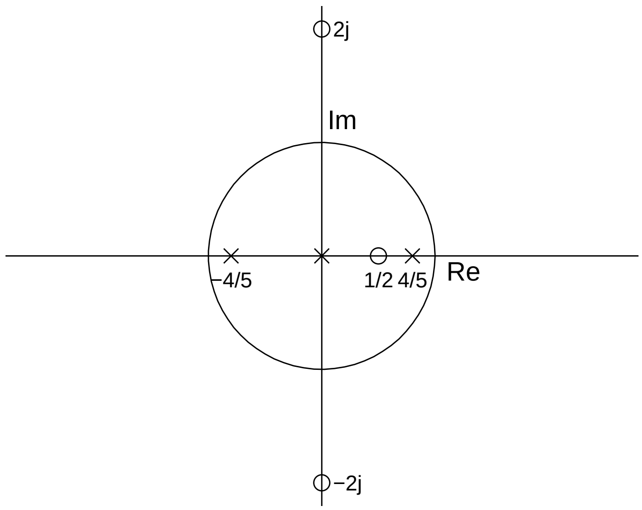
Thus, system A has generalized linear phase

$$B: \; (h_1[n] * h_2[n]) + h_3[n]$$

$h_1[n] * h_2[n]$ is symmetric about $n = 3$, as we noted above.
Adding $h_3[n]$ to this sequence will destroy all symmtery, so this does not have generalized linear phase.

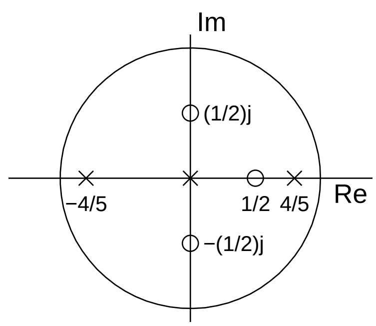
5.12.

$$H(z) = \frac{(1-0.5z^{-1})(1+2jz^{-1})(1-2jz^{-1})}{(1-0.8z^{-1})(1+0.8z^{-1})}$$

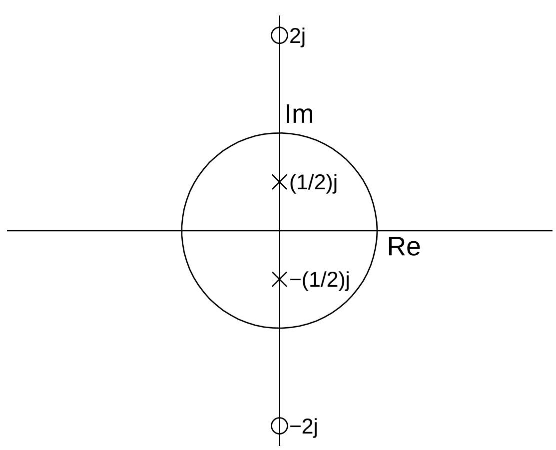


(a) A minimum phase system has all poles and zeros inside $|z| = 1$

$$H_1(z) = \frac{(1-0.5z^{-1})(1+\frac{1}{4}z^{-2})}{(1-0.64z^{-2})}$$

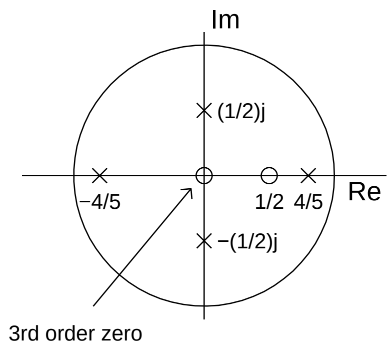


$$H_{ap}(z) = \frac{(1+4z^{-2})}{(1+\frac{1}{4}z^{-2})}$$

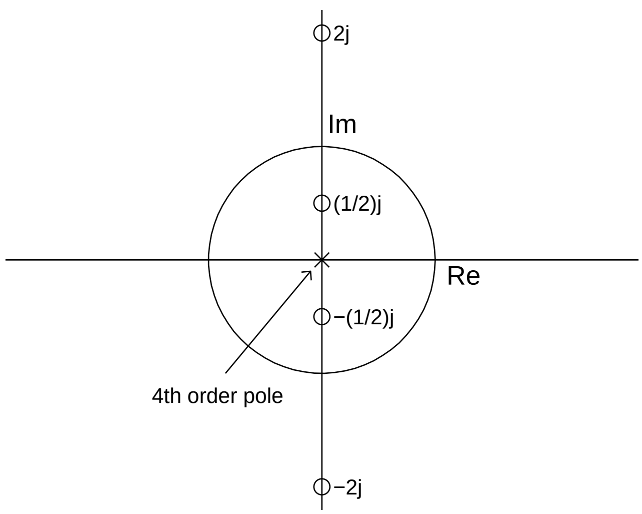


(b) A generalized linear phase system has zeros and poles at $z = 1, -1, 0$ or ∞ or in conjugate reciprocal pairs.

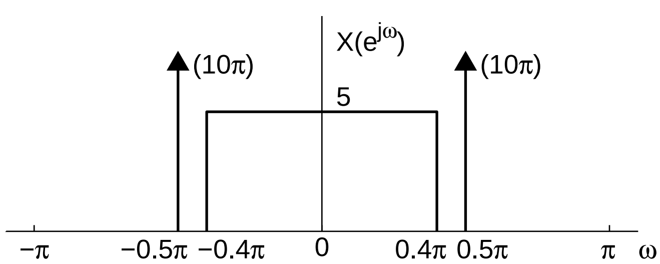
$$H_2(z) = \frac{(1 - 0.5z^{-1})}{(1 - 0.64z^{-2})(1 + \frac{1}{4}z^{-2})}$$



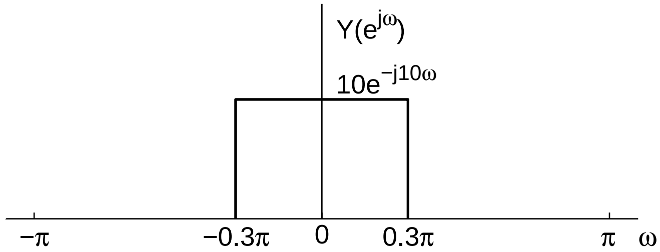
$$H_{lin}(z) = (1 + \tfrac{1}{4}z^{-2})(1 + 4z^{-2})$$



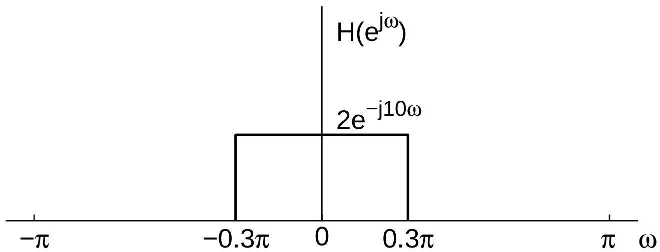
5.13. The input $x[n]$ in the frequency domain looks like



while the corresponding output $y[n]$ looks like



Therefore, the filter must be



In the time domain this is

$$h[n] = \frac{2\sin[0.3\pi(n-10)]}{\pi(n-10)}$$

5.14. (a)

Property	Applies?	Comments
Stable	No	For a stable, causal system, all poles must be inside the unit circle.
IIR	Yes	The system has poles at locations other than $z = 0$ or $z = \infty$.
FIR	No	FIR systems can only have poles at $z = 0$ or $z = \infty$.
Minimum Phase	No	Minimum phase systems have all poles and zeros located inside the unit circle.
Allpass	No	Allpass systems have poles and zeros in conjugate reciprocal pairs.
Generalized Linear Phase	No	The causal generalized linear phase systems presented in this chapter are FIR.
Positive Group Delay for all w	No	This system is not in the appropriate form.

(b)

Property	Applies?	Comments
Stable	Yes	The ROC for this system function, $\|z\| > 0$, contains the unit circle. (Note there is 7th order pole at $z = 0$).
IIR	No	The system has poles only at $z = 0$.
FIR	Yes	The system has poles only at $z = 0$.
Minimum Phase	No	By definition, a minimum phase system must have all its poles and zeros located *inside* the unit circle.
Allpass	No	Note that the zeros on the unit circle will cause the magnitude spectrum to drop zero at certain frequencies. Clearly, this system is not allpass.
Generalized Linear Phase	Yes	This is the pole/zero plot of a type II FIR linear phase system.
Positive Group Delay for all w	Yes	This system is causal and linear phase. Consequently, its group delay is a positive constant.

(c)

Property	Applies?	Comments
Stable	Yes	All poles are inside the unit circle. Since the system is causal, the ROC includes the unit circle.
IIR	Yes	The system has poles at locations other than $z = 0$ or $z = \infty$.
FIR	No	FIR systems can only have poles at $z = 0$ or $z = \infty$.
Minimum Phase	No	Minimum phase systems have all poles and zeros located inside the unit circle.
Allpass	Yes	The poles inside the unit circle have corresponding zeros located at conjugate reciprocal locations.
Generalized Linear Phase	No	The causal generalized linear phase systems presented in this chapter are FIR.
Positive Group Delay for all w	Yes	Stable allpass systems have positive group delay for all w.

5.15. (a) *Yes.*

By the region of convergence we know there are no poles at $z = \infty$ and it therefore must be causal. Another way to see this is to use long division to write $H_1(z)$ as

$$H_1(z) = \frac{1 - z^{-5}}{1 - z^{-1}} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}\ ,\ |z| > 0$$

(b) $h_1[n]$ is a causal rectangular pulse of length 5. If we convolve $h_1[n]$ with another causal rectangular pulse of length N we will get a triangular pulse of length $N + 5 - 1 = N + 4$. The triangular pulse is symmetric around its apex and thus has linear phase. To make the triangular pulse $g[n]$ have at least 9 nonzero samples we can choose $N = 5$ or let $h_2[n] = h_1[n]$.

Proof:

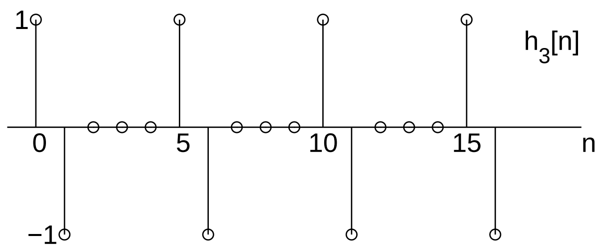
$$G(e^{j\omega}) \quad = \quad H_1(e^{j\omega})H_2(e^{j\omega}) = H_1^2(e^{j\omega})$$

$$
\begin{aligned}
&= \left[\frac{1-e^{-j5\omega}}{1-e^{-j\omega}}\right]^2 \\
&= \left[\frac{e^{-j\omega 5/2}\left(e^{j\omega 5/2}-e^{-j\omega 5/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2}-e^{-j\omega/2}\right)}\right]^2 \\
&= \frac{\sin^2(5\omega/2)}{\sin^2(\omega/2)}\, e^{-j4\omega}
\end{aligned}
$$

(c) The required values for $h_3[n]$ can intuitively be worked out using the flip and slide idea of convolution. Here is a second way to get the answer. Pick $h_3[n]$ to be the inverse system for $h_1[n]$ and then simplify using the geometric series as follows.

$$
\begin{aligned}
H_3(z) &= \frac{1-z^{-1}}{1-z^{-5}} \\
&= (1-z^{-1})\left[1+z^{-5}+z^{-10}+z^{-15}+\cdots\right] \\
&= 1-z^{-1}+z^{-5}-z^{-6}+z^{-10}-z^{-11}+z^{-15}-z^{-16}+\cdots
\end{aligned}
$$

This choice for $h_3[n]$ will make $q[n]=\delta[n]$ for all n. However, since we only need equality for $0 \le n \le 19$ truncating the infinite series will give us the desired result. The final answer is shown below.



5.16. (a) This system does not necessarily have generalized linear phase.

The phase response,

$$
G_1(e^{j\omega}) = \tan^{-1}\left(\frac{Im(H_1(e^{j\omega})+H_2(e^{j\omega}))}{Re(H_1(e^{j\omega})+H_2(e^{j\omega}))}\right)
$$

is not necessarily linear. As a counter-example, consider the systems

$$
\begin{aligned}
h_1[n] &= \delta[n]+\delta[n-1] \\
h_2[n] &= 2\delta[n]-2\delta[n-1] \\
g_1[n] &= h_1[n]+h_2[n] = 3\delta[n]-\delta[n-1] \\
G_1(e^{j\omega}) &= 3-e^{-j\omega} = 3-\cos\omega+j\sin\omega \\
\angle G_1(e^{j\omega}) &= \tan^{-1}\left(\frac{\sin\omega}{3-\cos\omega}\right)
\end{aligned}
$$

Clearly, $G_1(e^{j\omega})$ does not have linear phase.

(b) This system must have generalized linear phase.

$$
\begin{aligned}
G_2(e^{j\omega}) &= H_1(e^{j\omega})H_2(e^{j\omega}) \\
\left|G_2(e^{j\omega})\right| &= \left|H_1(e^{j\omega})\right|\left|H_2(e^{j\omega})\right| \\
\angle G_2(e^{j\omega}) &= \angle H_1(e^{j\omega}) + \angle H_2(e^{j\omega})
\end{aligned}
$$

The sum of two linear phase responses is also a linear phase response.

(c) This system does not necessarily have linear phase. Using properties of the DTFT, the circular convolution of $H_1(e^{jw})$ and $H_2(e^{jw})$ is related to the product of $h_1[n]$ and $h_2[n]$. Consider the systems

$$
\begin{aligned}
h_1[n] &= \delta[n] + \delta[n-1] \\
h_2[n] &= \delta[n] + 2\delta[n-1] + \delta[n-2] \\
g_3[n] &= h_1[n]h_2[n] = \delta[n] + 2\delta[n-1] \\
G_3(e^{j\omega}) &= 1 + 2e^{-j\omega} = 1 + 2\cos\omega - j2\sin\omega \\
\angle G_3(e^{j\omega}) &= \tan^{-1}\left(\frac{2\sin\omega}{1+2\cos\omega}\right)
\end{aligned}
$$

Clearly, $G_3(e^{j\omega})$ does not have linear phase.

5.17. For all of the following we know that the poles and zeros are real or occur in complex conjugate pairs since each impulse response is real. Since they are causal we also know that none have poles at infinity.

(a)
- Since $h_1[n]$ is real there are complex conjugate poles at $z = 0.9e^{\pm j\pi/3}$.
- If $x[n] = u[n]$

$$Y(z) = H_1(z)X(z) = \frac{H_1(z)}{1 - z^{-1}}$$

  We can perform a partial fraction expansion on $Y(z)$ and find a term $(1)^n u[n]$ due to the pole at $z = 1$. Since $y[n]$ eventually decays to zero this term must be cancelled by a zero. Thus, the filter must have a zero at $z = 1$.
- The length of the impulse response is infinite.

(b)
- Linear phase and a real impulse response implies that zeros occur at conjugate reciprocal locations so there are zeros at $z = z_1, 1/z_1, z_1^*, 1/z_1^*$ where $z_1 = 0.8e^{j\pi/4}$.
- Since $h_2[n]$ is both causal and linear phase it must be a Type I, II, III, or IV FIR filter. Therefore the filter's poles only occur at $z = 0$.
- Since the arg $\left\{H_2(e^{jw})\right\} = -2.5\omega$ we can narrow down the filter to a Type II or Type IV filter. This also tells us that the length of the impulse response is 6 and that there are 5 zeros. Since the number of poles always equal the number of zeros, we have 5 poles at $z = 0$.
- Since $20\log\left|H_2(e^{j0})\right| = -\infty$ we must have a zero at $z = 1$. This narrows down the filter type even more from a Type II or Type IV filter to just a Type IV filter.

With all the information above we can determine $H_2(z)$ completely (up to a scale factor)

$$H_2(z) = A(1 - z^{-1})(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})(1 - 1.25e^{j\pi/4}z^{-1})(1 - 1.25e^{-j\pi/4}z^{-1})$$

(c) Since $H_3(z)$ is allpass we know the poles and zeros occur in conjugate reciprocal locations. The impulse response is infinite and in general looks like

$$H_3(z) = \frac{(z^{-1} - 0.8e^{j\pi/4})(z^{-1} - 0.8e^{-j\pi/4})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})}H_{ap}(z)$$

5.18. (a) To be rational, X(z) must be of the form

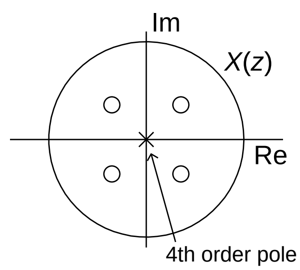
$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Because x[n] is real, its zeros must appear in conjugate pairs. Consequently, there are two more zeros, at $z = \frac{1}{2}e^{-j\pi/4}$, and $z = \frac{1}{2}e^{-j3\pi/4}$. Since x[n] is zero outside $0 \leq n \leq 4$, there are only four zeros (and poles) in the system function. Therefore, the system function can be written as

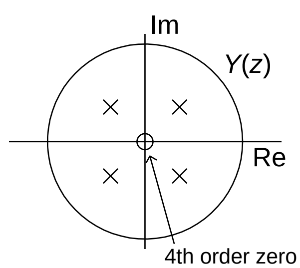
$$X(z) = \left(1 - \frac{1}{2}e^{j\pi/4}z^{-1}\right)\left(1 - \frac{1}{2}e^{j3\pi/4}z^{-1}\right)\left(1 - \frac{1}{2}e^{-j\pi/4}z^{-1}\right)\left(1 - \frac{1}{2}e^{-j3\pi/4}z^{-1}\right)$$

Clearly, X(z) is rational.

(b) A sketch of the pole-zero plot for $X(z)$ is shown below. Note that the ROC for X(z) is $|z| > 0$.



(c) A sketch of the pole-zero plot for $Y(z)$ is shown below. Note that the ROC for Y(z) is $|z| > \frac{1}{2}$.



5.19.

- Since $x[n]$ is real the poles & zeros come in complex conjugate pairs.
- From (1) we know there are no poles except at zero or infinity.
- From (3) and the fact that $x[n]$ is finite we know that the signal has generalized linear phase.
- From (3) and (4) we have $\alpha = 2$. This and the fact that there are no poles in the finite plane except the five at zero (deduced from (1) and (2)) tells us the form of $X(z)$ must be

$$X(z) = x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + x[4]z^{-4} + x[5]z^{-5}$$

  The phase changes by π at $\omega = 0$ and π so there must be a zero on the unit circle at $z = \pm 1$. The zero at $z = 1$ tells us $\sum x[n] = 0$. The zero at $z = -1$ tells us $\sum(-1)^n x[n] = 0$.
  We can also conclude $x[n]$ must be a Type III filter since the length of $x[n]$ is odd and there is a zero at both $z = \pm 1$. $x[n]$ must therefore be antisymmetric around $n = 2$ and $x[2] = 0$.
- From (5) and Parseval's theorem we have $\sum |x[n]|^2 = 28$.

- From (6)

$$\begin{aligned} y[0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})d\omega = 4 \\ &= x[n] * u[n]\,|_{n=0} = x[-1] + x[0] \end{aligned}$$

$$\begin{aligned} y[1] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{jw}d\omega = 6 \\ &= x[n] * u[n]\,|_{n=1} = x[-1] + x[0] + x[1] \end{aligned}$$

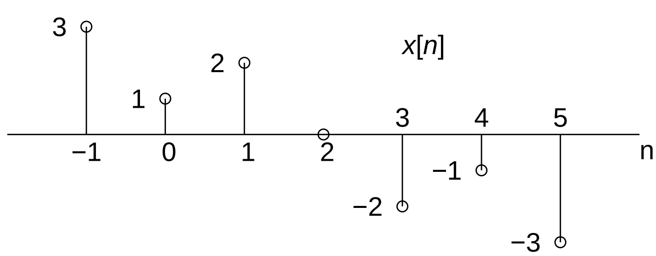
- The conclusion from (7) that $\sum(-1)^n x[n] = 0$ we already derived earlier.
- Since the DTFT $\{x_e[n]\} = \mathcal{R}e\left\{X(e^{j\omega})\right\}$ we have

$$\begin{aligned} \frac{x[5] + x[-5]}{2} &= -\frac{3}{2} \\ x[5] &= -3 + x[-5] \\ x[5] &= -3 \end{aligned}$$

Summarizing the above we have the following (dependent) equations

(1) $x[-1] + x[0] + x[1] + x[2] + x[3] + x[4] + x[5] = 0$
(2) $-x[-1] + x[0] - x[1] + x[2] - x[3] + x[4] - x[5] = 0$
(3) $x[2] = 0$
(4) $x[-1] = -x[5]$
(5) $x[0] = -x[4]$
(6) $x[1] = -x[3]$
(7) $x[-1]^2 + x[0]^2 + x[1]^2 + x[2]^2 + x[3]^2 + x[4]^2 + x[5]^2 = 28$
(8) $x[-1] + x[0] = 4$
(9) $x[-1] + x[0] + x[1] = 6$
(10) $x[5] = -3$

$x[n]$ is easily obtained from solving the equations in the following order: (3),(10),(4),(8),(5),(9), and (6).



Solutions – Chapter 6

Structures for Discrete-Time Systems

6.1. Causal LTI system with system function:

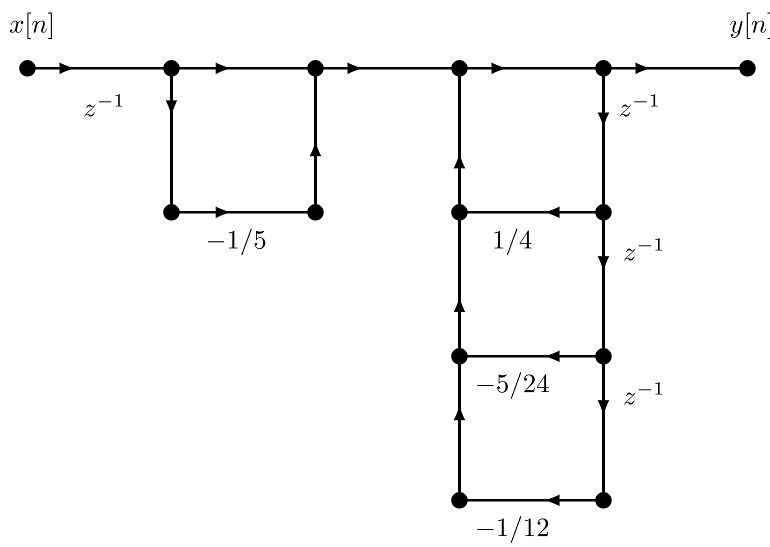
$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{(1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2})(1 + \frac{1}{4}z^{-1})}.$$

(a) (i) Direct form I.

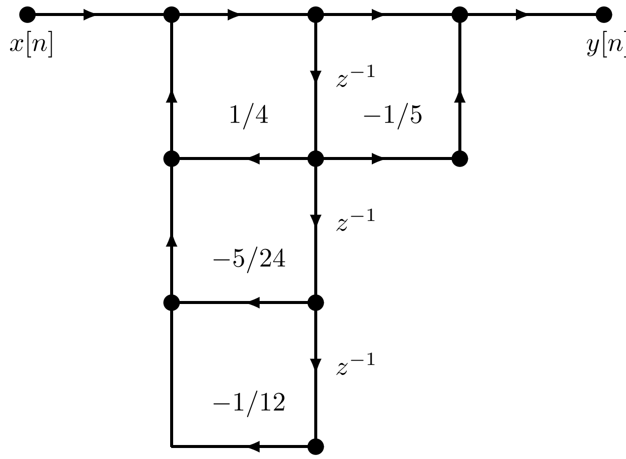
$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{1 - \frac{1}{4}z^{-1} + \frac{5}{24}z^{-2} + \frac{1}{12}z^{(}-3)}$$

so

$$b_0 = 1 \ , \ b_1 = -\frac{1}{5} \text{ and } a_1 = \frac{1}{4} \ , \ a_2 = -\frac{5}{24} \ , \ a_3 = -\frac{1}{12}.$$



(ii) Direct form II.

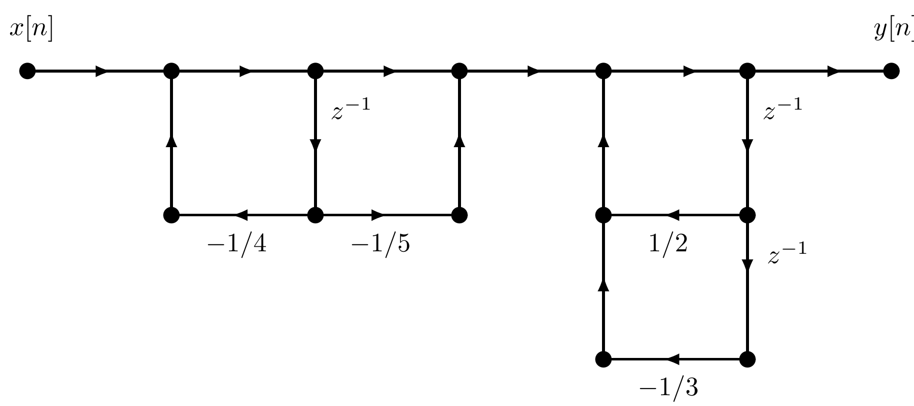


(iii) Cascade form using first and second order direct form II sections.

$$H(z) = \left(\frac{1 - \frac{1}{5}z^{-1}}{1 + \frac{1}{4}z^{-1}}\right)\left(\frac{1}{1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2}}\right).$$

So

$$b_{01} = 1 \ , \ b_{11} = -\tfrac{1}{5} \ , \ b_{21} = 0 \ ,$$
$$b_{02} = 1 \ , \ b_{12} = 0 \ , \ b_{22} = 0 \text{ and}$$
$$a_{11} = -\tfrac{1}{4} \ , \ a_{21} = 0 \ , \ a_{12} = \tfrac{1}{2} \ , \ a_{22} = -\tfrac{1}{3}.$$

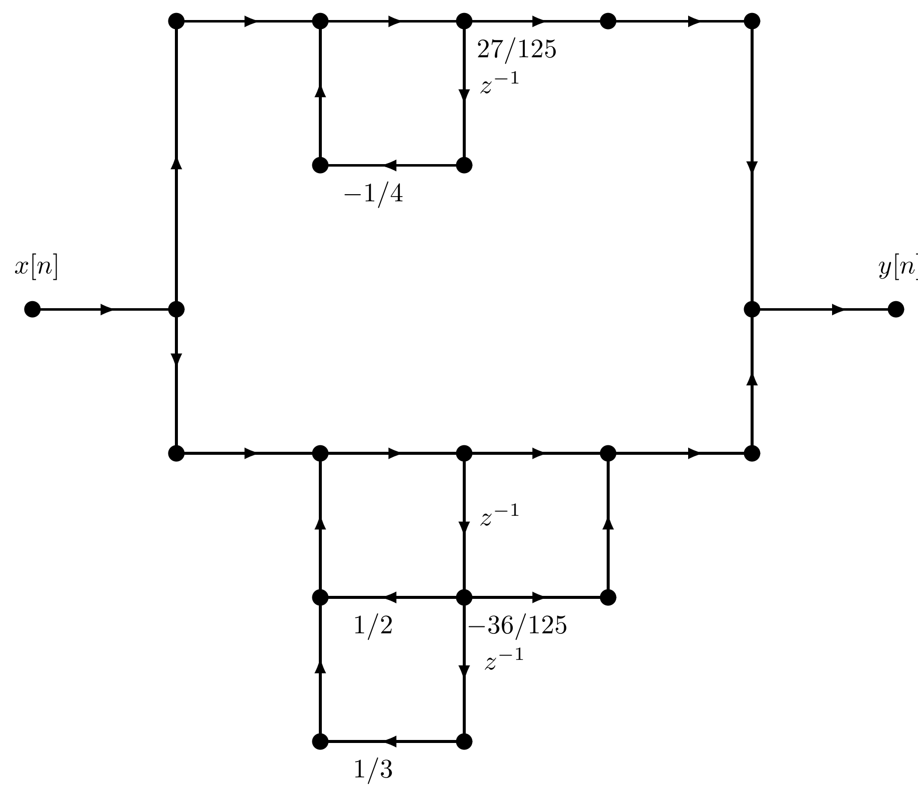


(iv) Parallel form using first and second order direct form II sections.
We can rewrite the transfer function as:

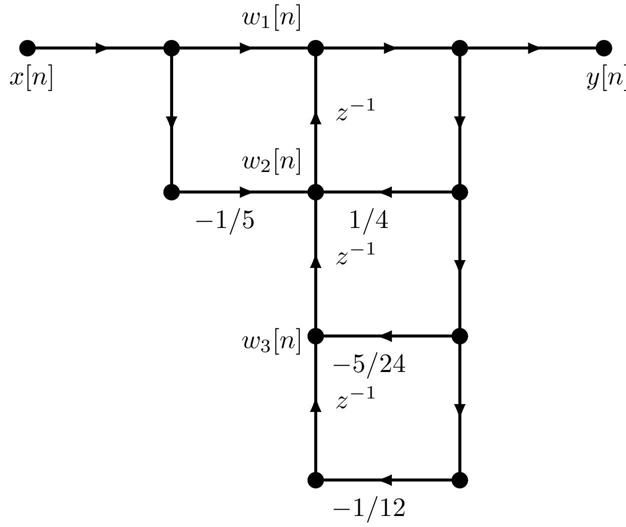
$$H(z) = \frac{\frac{27}{125}}{1 + \frac{1}{4}z^{-1}} + \frac{\frac{98}{125} - \frac{36}{125}z^{-1}}{1 - \frac{1}{2}z^{-1} - \frac{1}{3}z^{-2}}.$$

So

$$e_{01} = \frac{27}{125}\ ,\ e_{11} = 0\ ,$$
$$e_{02} = \frac{98}{125}\ ,\ e_{12} = -\frac{36}{125}\ ,\ \text{ and}$$
$$a_{11} = -\frac{1}{4}\ ,\ a_{21} = 0\ ,\ a_{12} = \frac{1}{2}\ ,\ a_{22} = -\frac{1}{3}.$$



(v) Transposed direct form II
We take the direct form II derived in part (ii) and reverse the arrows as well as exchange the input and output. Then redrawing the flow graph, we get:



(b) To get the difference equation for the flow graph of part (v) in (a), we first define the intermediate variables: $w_1[n]$, $w_2[n]$ and $w_3[n]$. We have:

$$\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= \frac{1}{4}y[n] + w_3[n-1] - \frac{1}{5}x[n] \\
(3)\ w_3[n] &= -\frac{5}{24}y[n] - \frac{1}{12}y[n-1] \\
(4)\quad y[n] &= w_1[n].
\end{aligned}$$

Combining the above equations, we get:

$$y[n] - \frac{1}{4}y[n-1] + \frac{5}{24}y[n-2] + \frac{1}{12}y[n-3] = x[n] - \frac{1}{5}x[n-1].$$

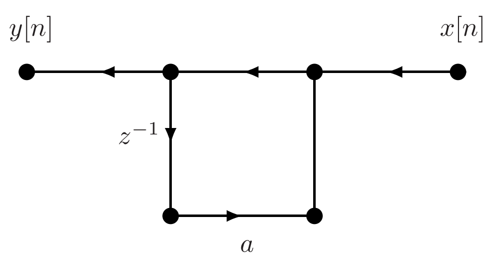
Taking the Z-transform of this equation and combining terms, we get the following transfer function:

$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{1 - \frac{1}{4}z^{-1} + \frac{5}{24}z^{-2} + \frac{1}{12}z^{-3}}$$

which is equal to the initial transfer function.

6.2. (a)

$$H(z) = \frac{1}{1 - az^{-1}}$$

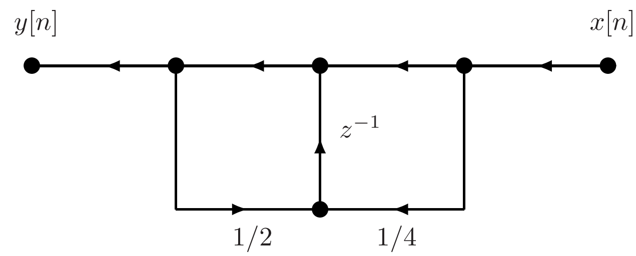


$$y[n] = x[n] + ay[n-1]$$

$$H_T(z) = \frac{1}{1 - az^{-1}} = H(z)$$

(b)

$$H(z) = \frac{1 + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

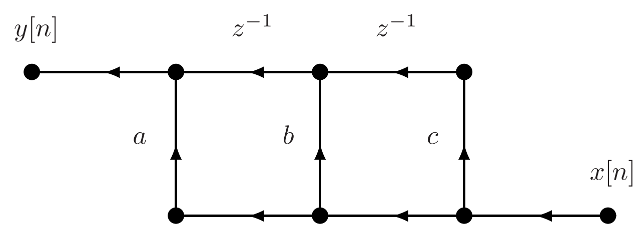


$$y[n] = x[n] + \frac{1}{4}x[n-1] + \frac{1}{2}y[n-1]$$

$$H_T(z) = \frac{1 + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}} = H(z)$$

(c)

$$H(z) = a + bz^{-1} + cz^{-2}$$

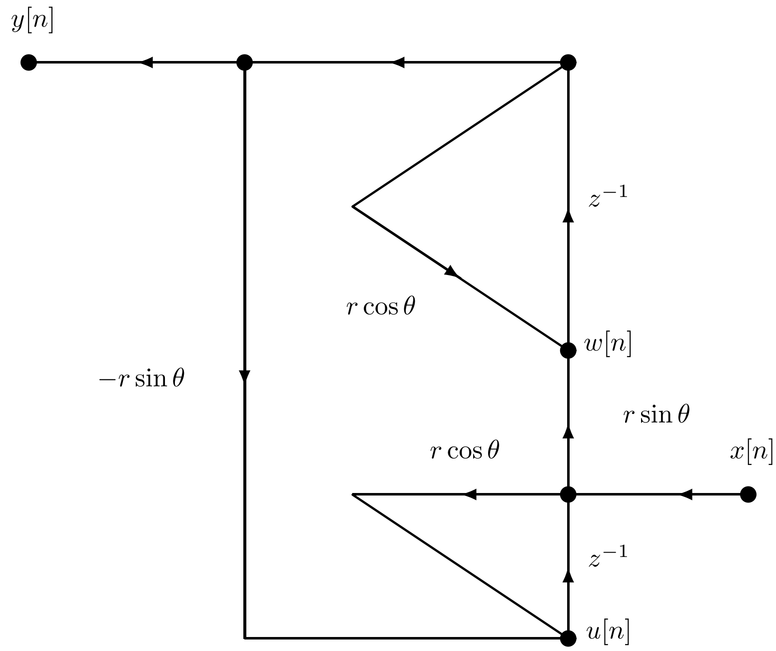


$$y[n] = ax[n] + bx[n-1] + cx[n-2]$$

$$H_T(z) = a + bz^{-1} + cz^{-2} = H(z)$$

(d)

$$H(z) = \frac{r \sin\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$



$$\begin{aligned}
V &= X + z^{-1}U \\
U &= r\cos\theta V - r\sin\theta Y \\
W &= r\sin\theta V + r\cos\theta z^{-1}W \\
Y &= z^{-1}W \\
\Longrightarrow \frac{Y}{X} &= H_T(z) \\
&= \frac{r\sin\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}} \\
&= H(z).
\end{aligned}$$

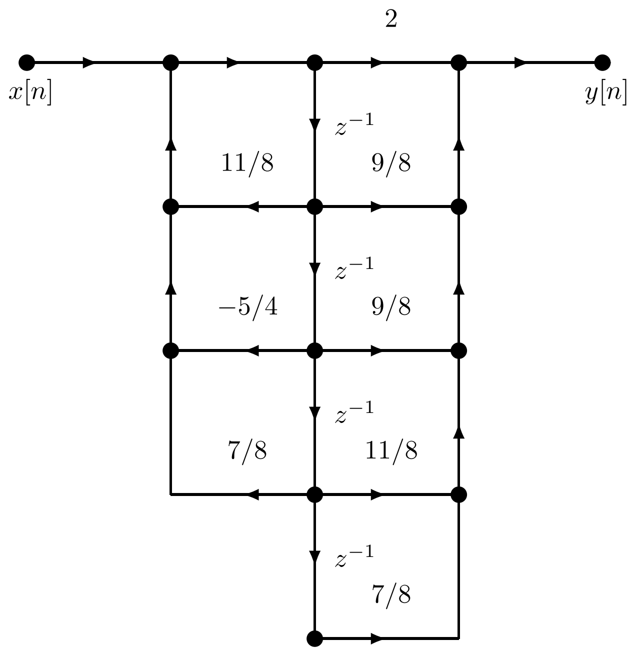
6.3. (a)

$$\begin{aligned}
H(z) &= \frac{1}{1-z^{-1}}\left[\frac{1-\frac{1}{2}z^{-1}}{1-\frac{3}{8}z^{-1}+\frac{7}{8}z^{-2}} + 1 + 2z^{-1} + z^{-2}\right] \\
&= \frac{2+\frac{9}{8}z^{-1}+\frac{9}{8}z^{-2}+\frac{11}{8}z^{-3}+\frac{7}{8}z^{-4}}{1-\frac{11}{8}z^{-1}+\frac{5}{4}z^{-2}-\frac{7}{8}z^{-3}}.
\end{aligned}$$

(b)

$$\begin{aligned}
y[n] &= 2x[n] + \frac{9}{8}x[n-1] + \frac{9}{8}x[n-2] + \frac{11}{8}x[n-3] + \frac{7}{8}x[n-4] \\
&+ \frac{11}{8}y[n-1] - \frac{5}{4}y[n-2] + \frac{7}{8}y[n-3].
\end{aligned}$$

(c) Use Direct Form II:



2
$x[n]$
$y[n]$
z^{-1}
11/8
9/8
z^{-1}
−5/4
9/8
z^{-1}
7/8
11/8
z^{-1}
7/8

Solutions – Chapter 7

Filter Design Techniques

7.1. (a) Using the bilinear transform frequency mapping equation,

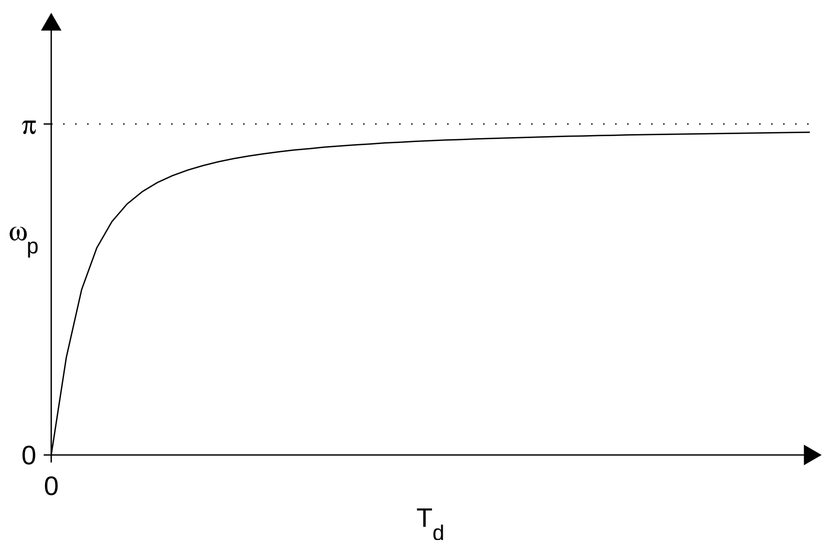
$$\Omega_p = \frac{2}{T_d} \tan\left(\frac{\omega_p}{2}\right)$$

we have

$$\begin{aligned} T_d &= \frac{2}{\Omega_p} \tan\left(\frac{\pi}{4}\right) \\ &= \frac{2}{\Omega_p} \end{aligned}$$

(b)

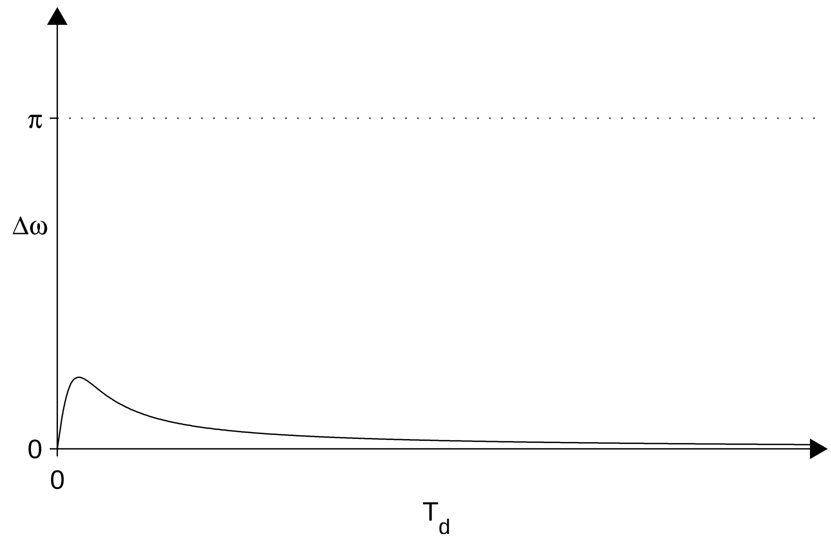
$$\omega_p = 2 \tan^{-1}\left(\frac{\Omega_p T_d}{2}\right)$$



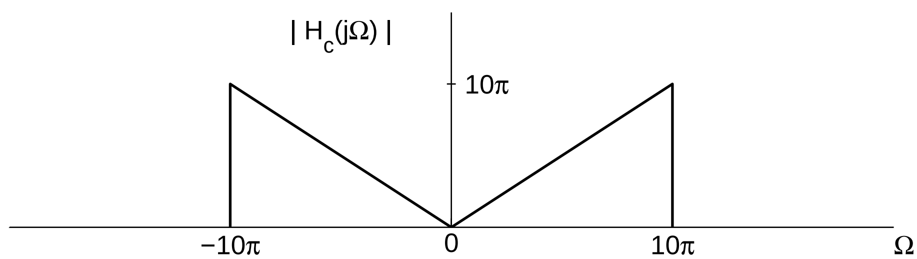
(c)

$$\begin{aligned} \omega_s &= 2 \tan^{-1}\left(\frac{\Omega_s T_d}{2}\right) \\ \omega_p &= 2 \tan^{-1}\left(\frac{\Omega_p T_d}{2}\right) \end{aligned}$$

$$\Delta\omega = \omega_s - \omega_p = 2\left[\tan^{-1}\left(\frac{\Omega_s T_d}{2}\right) - \tan^{-1}\left(\frac{\Omega_p T_d}{2}\right)\right]$$

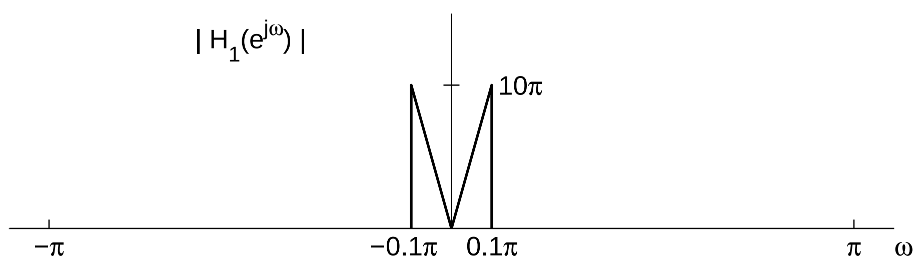


7.2. We start with $|H_c(j\Omega)|$,



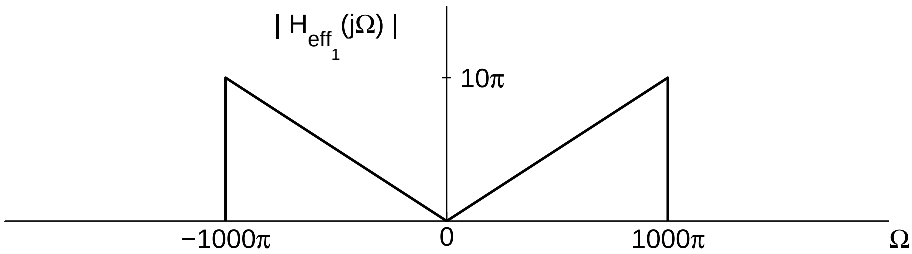
(a) By impulse invariance we scale the frequency axis by T_d to get

$$|H_1(e^{j\omega})| = \left| \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi k}{T_d}\right) \right|$$



Then, to get the overall system response we scale the frequency axis by T and bandlimit the result according to the equation

$$|H_{\mathrm{eff}_1}(j\Omega)| = \begin{cases} |H_1(e^{j\omega T})|, & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}$$

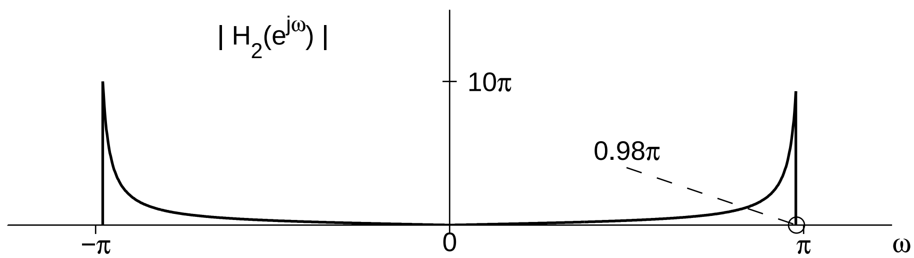


(b) Using the frequency mapping relationships of the bilinear transform,

$$\begin{aligned}\Omega &= \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right),\\ \omega &= 2\tan^{-1}\left(\frac{\Omega T_d}{2}\right),\end{aligned}$$

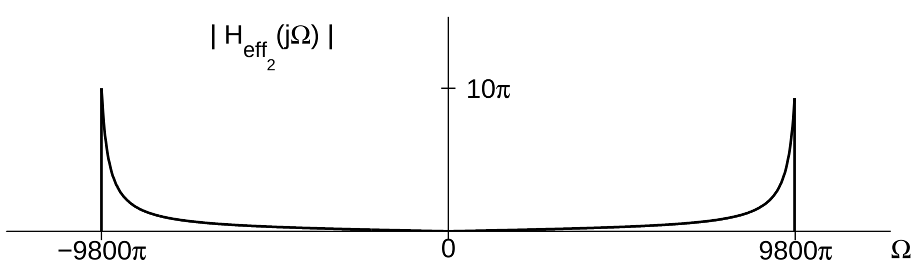
we get

$$|H_2(e^{j\omega})| = \begin{cases} |\tan\left(\frac{\omega}{2}\right)|, & |\omega| < 2\tan^{-1}(10\pi) = 0.98\pi \\ 0, & \text{otherwise}\end{cases}$$



Then, to get the overall system response we scale the frequency axis by T and bandlimit the result according to the equation

$$|H_{\text{eff}_2}(j\Omega)| = \begin{cases} |H_2(e^{j\omega T})|, & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T}\end{cases}$$



7.3. (a) Since

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right)$$

and we desire

$$H(e^{j\omega})\mid_{\omega=0} = H_c(j\Omega)\mid_{\Omega=0},$$

we see that

$$H(e^{j\omega})|_{\omega=0} = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right)|_{\omega=0} = H_c(j\Omega)|_{\Omega=0}$$

requires

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = 0.$$

(b) Since the bilinear transform maps $\Omega = 0$ to $\omega = 0$, the condition will hold for any choice of $H_c(j\Omega)$.

7.4. (a)

$$s = \frac{1+z^{-1}}{1-z^{-1}} \longleftrightarrow z = \frac{s+1}{s-1}$$

Now, we evaluate the above expressions along the $j\Omega$ axis of the s-plane

$$\begin{aligned} z &= \frac{j\Omega+1}{j\Omega-1} \\ |z| &= 1 \end{aligned}$$

(b) We want to show $|z| < 1$ if $\mathcal{R}e\{s\} < 0$.

$$\begin{aligned} z &= \frac{\sigma+j\Omega+1}{\sigma+j\Omega-1} \\ |z| &= \frac{\sqrt{(\sigma+1)^2+\Omega^2}}{\sqrt{(\sigma-1)^2+\Omega^2}} \end{aligned}$$

Therefore, if $|z| < 1$

$$\begin{aligned} (\sigma+1)^2+\Omega^2 &< (\sigma-1)^2+\Omega^2 \\ \sigma &< -\sigma \end{aligned}$$

it must also be true that $\sigma < 0$. We have just shown that the left-half s-plane maps to the interior of the z-plane unit circle. Thus, any pole of $H_c(s)$ inside the left-half s-plane will get mapped to a pole inside the z-plane unit circle.

(c) We have the relationship

$$\begin{aligned} j\Omega &= \frac{1+e^{-j\omega}}{1-e^{-j\omega}} \\ &= \frac{e^{j\omega/2}+e^{-j\omega/2}}{e^{j\omega/2}-e^{-j\omega/2}} \\ \Omega &= -\cot(\omega/2) \end{aligned}$$

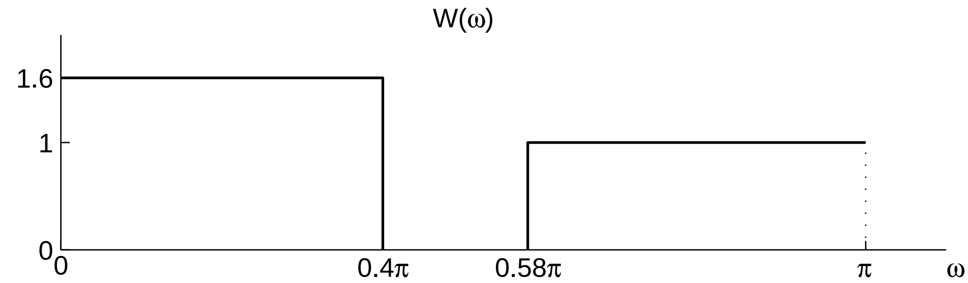
$$\begin{aligned} |\Omega_s| &= |\cot(\pi/6)| = \sqrt{3} \\ |\Omega_{p_1}| &= |\cot(\pi/2)| = 0 \\ |\Omega_{p_2}| &= |\cot(\pi/4)| = 1 \end{aligned}$$

Therefore, the constraints are

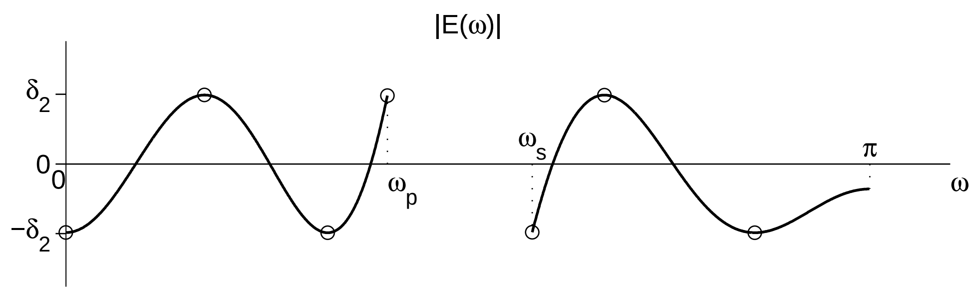
$$\begin{aligned} 0.95 \leq |H_c(j\Omega)| \leq 1.05, &\qquad 0 \leq |\Omega| \leq 1 \\ |H_c(j\Omega)| \leq 0.01, &\qquad \sqrt{3} \leq |\Omega| \end{aligned}$$

7.5. (a) Since $H(e^{j0}) \neq 0$ and $H(e^{j\pi}) \neq 0$, this must be a Type I filter.

(b) With the weighting in the stopband equal to 1, the weighting in the passband is $\frac{\delta_2}{\delta_1}$.



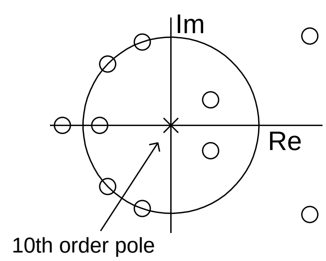
(c)



(d) An optimal (in the Parks-McClellan sense) Type I lowpass filter can have either $L+2$ or $L+3$ alternations. The second case is true only when an alternation occurs at all band edges. Since this filter does not have an alternation at $\omega = \pi$ it should only have $L+2$ alternations. From the figure, we see that there are 7 alternations so $L = 5$. Thus, the filter length is $2L+1 = 11$ samples long.

(e) Since the filter is 11 samples long, it has a delay of 5 samples.

(f) Note the zeroes off the unit circle are implied by the dips in the frequency response at the indicated frequencies.



7.6. *True*. Since filter C is a stable IIR filter it has poles in the left half plane. The bilinear transform maps the left half plane to the inside of the unit circle. Thus, the discrete filter B has to have poles and is therefore an IIR filter.

Solutions – Chapter 8

The Discrete-Time Fourier Transform

8.1. For a finite-length sequence $x[n]$, with length equal to N, the periodic repetition of $x[-n]$ is represented by

$$x[((-n))_N] = x[((-n + \ell N))_N], \qquad \ell\text{: integer}$$

where the right side is justified since $x[n]$ (and $x[-n]$) is periodic with period N.

The above statement holds true for any choice of ℓ. Therefore, for $\ell = 1$:

$$x[((-n))_n] = x[((-n + N))_N]$$

8.2. No. Recall that the DFT merely samples the frequency spectra. Therefore, the fact the $Im\{X[k]\} = 0$ for $0 \leq k \leq (N-1)$ does not guarantee that the imaginary part of the continuous frequency spectra is also zero.

For example, consider a signal which consists of an impulse centered at $n = 1$.

$$x[n] = \delta[n-1], \qquad 0 \leq n \leq 1$$

The Fourier transform is:

$$\begin{aligned} X(e^{j\omega}) &= e^{-j\omega} \\ Re\{X(e^{j\omega})\} &= \cos(\omega) \\ Im\{X(e^{j\omega})\} &= -\sin(\omega) \end{aligned}$$

Note that neither is zero for all $0 \leq \omega \leq 2$. Now, suppose we take the 2-pt DFT:

$$\begin{aligned} X[k] &= W_2^k, \qquad 0 \leq k \leq 1 \\ &= \begin{cases} 1, & k = 0 \\ -1, & k = 1 \end{cases} \end{aligned}$$

So, $Im\{X[k]\} = 0, \quad \forall k$. However, $Im\{X(e^{j\omega})\} \neq 0$.

Note also that the size of the DFT plays a large role. For instance, consider taking the 3-pt DFT of

$$\begin{aligned} x[n] &= \delta[n-1], \qquad 0 \leq n \leq 2 \\ X[k] &= W_3^k, \qquad 0 \leq k \leq 2 \\ &= \begin{cases} 1, & k = 0 \\ e^{-j(2\pi/3)}, & k = 1 \\ e^{-j(4\pi/3)}, & k = 2 \end{cases} \end{aligned}$$

Now, $Im\{X[k]\} \neq 0$, for $k = 1$ or $k = 2$.

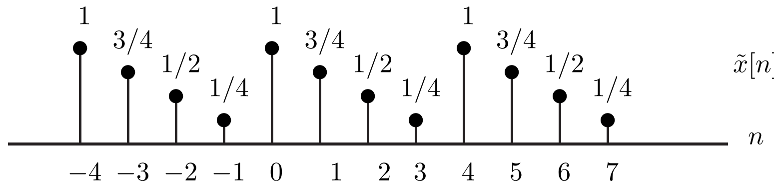
8.3. Both sequences $x[n]$ and $y[n]$ are of finite-length ($N = 4$).

Hence, no aliasing takes place. From Section 8.6.2, multiplication of the DFT of a sequence by a complex exponential corresponds to a circular shift of the time-domain sequence.

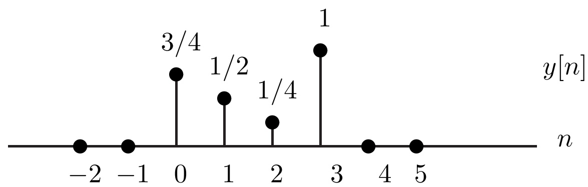
Given $Y[k] = W_4^{3k} X[k]$, we have

$$y[n] = x[((n-3))_4]$$

We use the technique suggested in problem 8.28. That is, we temporarily extend the sequence such that a periodic sequence with period 4 is formed.



Now, we shift by three (to the right), and set all values outside $0 \le n \le 3$ to zero.



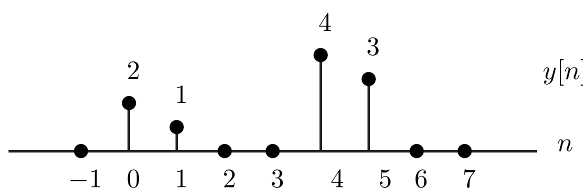
8.4. (a) When multiplying the DFT of a sequence by a complex exponential, the time-domain signal undergoes a circular shift.

For this case,

$$Y[k] = W_6^{4k} X[k], \qquad 0 \le k \le 5$$

Therefore,

$$y[n] = x[((n-4))_6], \qquad 0 \le n \le 5$$



(b) There are two ways to approach this problem. First, we attempt a solution by brute force.

$$\begin{aligned}
X[k] &= 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k}, \qquad W_6^k = e^{-j(2\pi k/6)} \text{ and } 0 \le k \le 5 \\
W[k] &= \mathcal{R}e\{X[k]\} \\
&= \frac{1}{2}\left(X[k] + X^*[k]\right) \\
&= \frac{1}{2}\left(4 + 3W_6^k + 2W_6^{2k} + W_6^{3k} + 4 + 3W_6^{-k} + 2W_6^{-2k} + W_6^{-3k}\right)
\end{aligned}$$

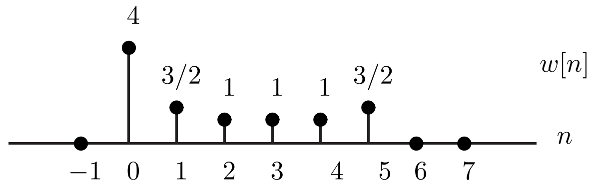
Notice that

$$\begin{aligned}
W_N^k &= e^{-j(2\pi k/N)} \\
W_N^{-k} &= e^{j(2\pi k/N)} = e^{-j(2\pi/N)(N-k)} = W_N^{N-k} \\
W[k] &= 4 + \frac{3}{2}\left[W_6^k + W_6^{6-k}\right] + \left[W_6^{2k} + W_6^{6-2k}\right] + \frac{1}{2}\left[W_6^{3k} + W_6^{6-3k}\right], \qquad 0 \le k \le 5
\end{aligned}$$

So,

$$\begin{aligned}
w[n] &= 4\delta[n] + \frac{3}{2}\Big(\delta[n-1] + \delta[n-5]\Big) + \delta[n-2] + \delta[n-4] \\
&\quad + \frac{1}{2}\Big(\delta[n-3] + \delta[n-3]\Big) \\
w[n] &= 4\delta[n] + \frac{3}{2}\delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \frac{3}{2}\delta[n-5], \qquad 0 \le n \le 5
\end{aligned}$$

Sketching $w[n]$:

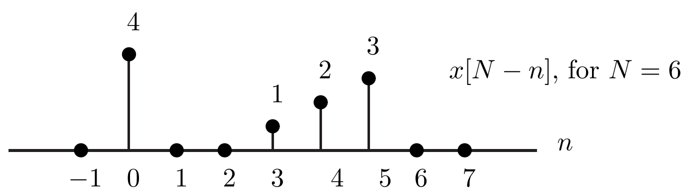


As an alternate approach, suppose we use the properties of the DFT as listed in Table 8.2.

$$\begin{aligned} W[k] &= \mathcal{R}e\{X[k]\} \\ &= \frac{X[k]+X^*[k]}{2} \\ w[n] &= \frac{1}{2}\ \text{IDFT}\{X[k]\} + \frac{1}{2}\ \text{IDFT}\{X^*[k]\} \\ &= \frac{1}{2}\Big(x[n] + x^*[((-n))_N]\Big) \end{aligned}$$

For $0 \le n \le N-1$ and $x[n]$ real:

$$w[n] = \frac{1}{2}\Big(x[n] + x[N-n]\Big)$$



So, we observe that $w[n]$ results as above.

(c) The DFT is decimated by two. By taking alternate points of the DFT output, we have half as many points. The influence of this action in the time domain is, as expected, the appearance of aliasing. For the case of decimation by two, we shall find that an additional replica of x[n] surfaces, since the sequence is now periodic with period 3.

From part (b):

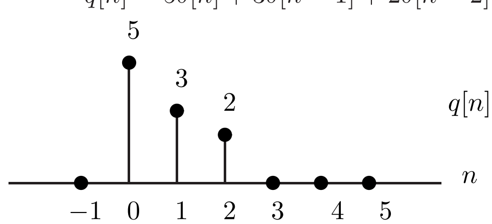
$$X[k] = 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k}, \qquad 0 \le k \le 5$$

Let $Q[k] = X[2k]$,

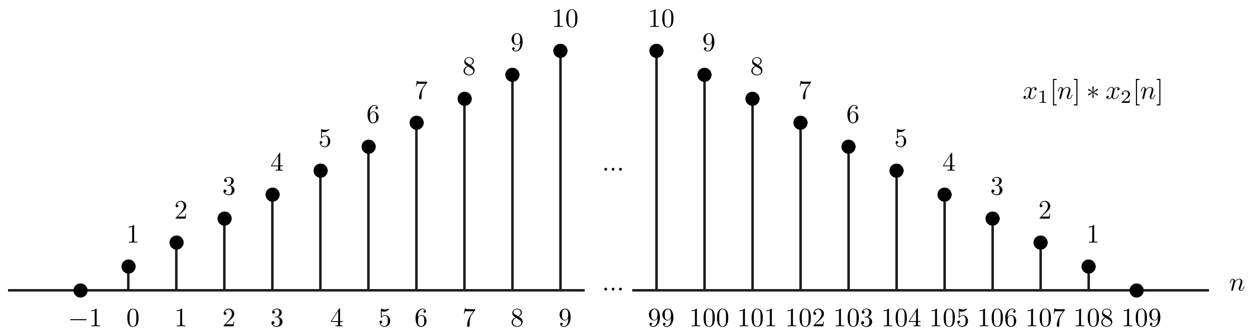
$$Q[k] = 4 + 3W_3^k + 2W_3^{2k} + W_3^{3k}, \qquad 0 \le k \le 2$$

Noting that $W_3^{3k} = W_3^{0k}$

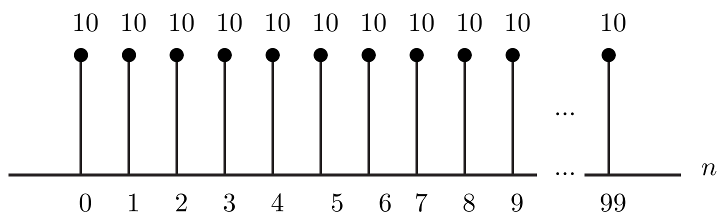
$$q[n] = 5\delta[n] + 3\delta[n-1] + 2\delta[n-2], \qquad 0 \le n \le 2$$



8.5. (a) The linear convolution, $x_1[n] * x_2[n]$ is a sequence of length $100 + 10 - 1 = 109$.

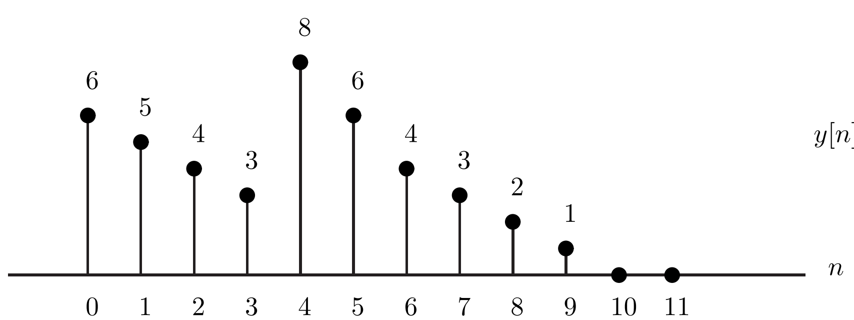


(b) The circular convolution, $x_1[n]\,100\,x_2[n]$, can be obtained by aliasing the first 9 points of the linear convolution above:



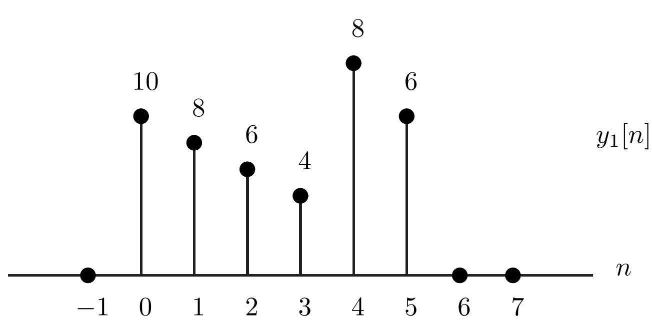
(c) Since $N \geq 109$, the circular convolution $x_1[n]\,110\,x_2[n]$ will be equivalent to the linear convolution of part (a).

8.6. Circular convolution equals linear convolution plus aliasing. First, we find $y[n] = x_1[n] * x_2[n]$:



Note that $y[n]$ is a ten point sequence ($N = 6 + 5 - 1$).

(a) For $N = 6$, the last four non-zero point ($6 \leq n \leq 9$) will alias to the first four points, giving us $y_1[n] = x_1[n]\,6\,x_2[n]$



(b) For $N = 10, N \geq 6 + 5 - 1$, so no aliasing occurs, and circular convolution is identical to linear convolution.

8.7. We have a finite length sequence, whose 64-pt DFT contains only one nonzero point (for $k = 32$).

(a) Using the synthesis equation Eq. (8.68):

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \qquad 0 \leq n \leq (N-1)$$

Substitution yields:

$$\begin{aligned}
x[n] &= \frac{1}{64}X[32]W_{64}^{-32n} \\
&= \frac{1}{64}e^{j\frac{2\pi}{64}(32)n} \\
&= \frac{1}{64}e^{j\pi n} \\
x[n] &= \frac{1}{64}(-1)^n, \qquad 0 \leq n \leq (N-1)
\end{aligned}$$

The answer is unique because we have taken the 64-pt DFT of a 64-pt sequence.

(b) The sequence length is now $N = 192$.

$$\begin{aligned}
x[n] &= \frac{1}{192}\sum_{k=0}^{191} X[k]W_{192}^{-kn}, \; 0 \leq n \leq 191 \\
x[n] &= \begin{cases} \frac{1}{64}(-1)^n & 0 \leq n \leq 63 \\ 0 & 64 \leq n \leq 191 \end{cases}
\end{aligned}$$

This solution is not unique. By taking only 64 spectral samples, $x[n]$ will be aliased in time. As an alternate sequence, consider

$$x'[n] = \frac{1}{64}\left(\frac{1}{3}\right)(-1)^n, \qquad 0 \leq n \leq 191$$

We have a finite-length sequence, $x[n]$ with $N = 8$. Suppose we interpolate by a factor of two. That is, we wish to double the size of $x[n]$ by inserting zeros at all odd values of n for $0 \leq n \leq 15$.

Mathematically,

$$y[n] = \begin{cases} x[n/2], & n \text{ even}, \quad 0 \leq n \leq 15 \\ 0, & n \text{ odd}, \end{cases}$$

The 16-pt. DFT of $y[n]$:

$$\begin{aligned}
Y[k] &= \sum_{n=0}^{15} y[n]W_{16}^{kn}, \qquad 0 \leq k \leq 15 \\
&= \sum_{n=0}^{7} x[n]W_{16}^{2kn}
\end{aligned}$$

Recall, $W_{16}^{2kn} = e^{j(2\pi/16)(2k)n} = e^{-j(2\pi/8)kn} = W_8^{kn}$,

$$Y[k] = \sum_{n=0}^{7} x[n]W_8^{kn}, \qquad 0 \leq k \leq 15$$

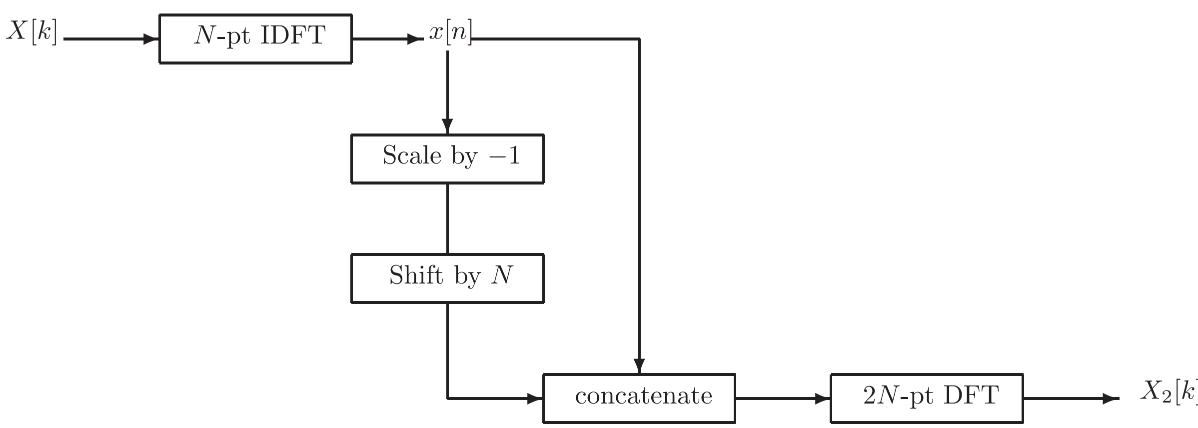
Therefore, the 16-pt. DFT of the interpolated signal contains two copies of the 8-pt. DFT of $x[n]$. This is expected since $Y[k]$ is now periodic with period 8 (see problem 8.1). Therefore, the correct choice is C.

As a quick check, $Y[0] = X[0]$.

8.9. (a) Since

$$x_2[n] = \begin{cases} x[n], & 0 \leq n \leq N-1 \\ -x[n-N], & N \leq n \leq 2N-1 \\ 0, & \text{otherwise} \end{cases}$$

If $X[k]$ is known, $x_2[n]$ can be constructed by :

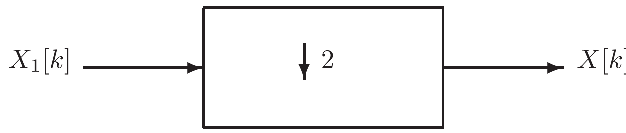


(b) To obtain $X[k]$ from $X_1[k]$, we might try to take the inverse DFT (2N-pt) of $X_1[k]$, then take the N-pt DFT of $x_1[n]$ to get $X[k]$.

However, the above approach is highly inefficient. A more reasonable approach may be achieved if we examine the DFT analysis equations involved. First,

$$\begin{aligned} X_1[k] &= \sum_{n=0}^{2N-1} x_1[n]W_{2N}^{kn}, \qquad 0 \leq k \leq (2N-1) \\ &= \sum_{n=0}^{N-1} x[n]W_{2N}^{kn} \\ &= \sum_{n=0}^{N-1} x[n]W_N^{(k/2)n}, \qquad 0 \leq k \leq (N-1) \\ X_1[k] &= X[k/2], \qquad 0 \leq k \leq (N-1) \end{aligned}$$

Thus, an easier way to obtain $X[k]$ from $X_1[k]$ is simply to decimate $X_1[k]$ by two.



8.10. (a) The DFT of the even part of a real sequence:

If $x[n]$ is of length N, then $x_e[n]$ is of length $2N-1$:

$$
\begin{aligned}
x_e[n] &= \begin{cases} \dfrac{x[n]+x[-n]}{2}, & (-N+1) \leq n \leq (N-1) \\ 0 & \text{otherwise} \end{cases} \\
X_e[k] &= \sum_{n=-N+1}^{N-1} \left(\frac{x[n]+x[-n]}{2} \right) W_{2N-1}^{kn}, \qquad (-N+1) \leq k \leq (N-1) \\
&= \sum_{n=-N+1}^{0} \frac{x[-n]}{2} W_{2N-1}^{kn} + \sum_{n=0}^{N-1} \frac{x[n]}{2} W_{2N-1}^{kn}
\end{aligned}
$$

Let $m = -n$,

$$
\begin{aligned}
X_e[k] &= \sum_{n=0}^{N-1} \frac{x[n]}{2} W_{2N-1}^{-kn} + \sum_{n=0}^{N-1} \frac{x[n]}{2} W_{2N-1}^{kn} \\
X_e[k] &= \sum_{n=0}^{N-1} x[n] \cos\left(\frac{2\pi kn}{2N-1} \right)
\end{aligned}
$$

Recall

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad 0 \leq k \leq (N-1)$$

and

$$Re\{X[k]\} = \sum_{n=0}^{N-1} x[n] \cos\left(\frac{2\pi kn}{N} \right)$$

So: $\text{DFT}\{x_e[n]\} \neq Re\{X[k]\}$

(b)

$$
\begin{aligned}
Re\{X[k]\} &= \frac{X[k]+X^*[k]}{2} \\
&= \frac{1}{2} \sum_{n=0}^{N-1} x[n] W_N^{kn} + \frac{1}{2} \sum_{n=0}^{N-1} x[n] W_N^{-kn} \\
&= \frac{1}{2} \sum_{n=0}^{N-1} (x[n] + x[N-n]) W_N^{kn}
\end{aligned}
$$

So,

$$Re\{X[k]\} = \text{DFT}\left\{ \frac{1}{2}(x[n] + x[N-n]) \right\}$$

8.11. From condition 1, we can determine that the sequence is of finite length ($N = 5$). Given:

$$\begin{aligned} X(e^{j\omega}) &= 1 + A_1 \cos\omega + A_2 \cos 2\omega \\ &= 1 + \frac{A_1}{2}(e^{j\omega} + e^{-j\omega}) + \frac{A_2}{2}(e^{j2\omega} + e^{-j2\omega}) \end{aligned}$$

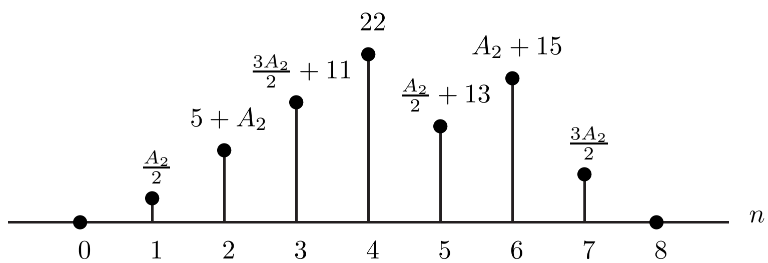
From the Fourier analysis equation, we can see by matching terms that:

$$x[n] = \delta[n] + \frac{A_1}{2}(\delta[n-1] + \delta[n+1]) + \frac{A_2}{2}(\delta[n-2] + \delta[n+2])$$

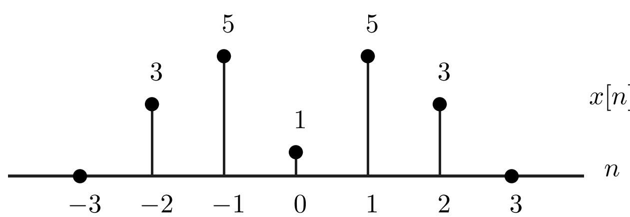
Condition 2 yields one of the values for the amplitude constants of condition 1. Since $x[n] * \delta[n-3] = x[n-3] = 5$ for $n = 2$, we know $x[-1] = 5$, and also that $x[1] = x[-1] = 5$. Knowing both these values tells us that $A_1 = 10$.

For condition 3, we perform a circular convolution between $x[((n-3))_8]$ and $w[n]$, a three-point sequence. For this case, linear convolution is the same as circular convolution since $N = 8 \geq 6 + 3 - 1$.

We know $x[((n-3))_8] = x[n-3]$, and convolving this with $w[n]$ from Fig P8.35-1 gives:



For $n = 2, w[n] * x[n-3] = 11$ so $A_2 = 6$. Thus, $x[2] = x[-2] = 3$, and we have fully specified $x[n]$:



8.12. We have the finite-length sequence:

$$x[n] = 2\delta[n] + \delta[n-1] + \delta[n-3]$$

(i) Suppose we perform the 5-pt DFT:

$$X[k] = 2 + W_5^k + W_5^{3k}, \qquad 0 \leq k \leq 5$$

where $W_5^k = e^{-j(\frac{2\pi}{5})k}$.

(ii) Now, we square the DFT of $x[n]$:

$$\begin{aligned} Y[k] &= X^2[k] \\ &= 2 + 2W_5^k + 2W_5^{3k} \\ &\quad + 2W_5^k + W_5^{2k} + W_5^{5k} \\ &\quad + 2W_5^{3k} + W_5^{4k} + W_5^{6k}, \qquad 0 \le k \le 5 \end{aligned}$$

Using the fact $W_5^{5k} = W_5^0 = 1$ and $W_5^{6k} = W_5^k$

$$Y[k] = 3 + 5W_5^k + W_5^{2k} + 4W_5^{3k} + W_5^{4k}, \qquad 0 \le k \le 5$$

(a) By inspection,

$$y[n] = 3\delta[n] + 5\delta[n-1] + \delta[n-2] + 4\delta[n-3] + \delta[n-4], \qquad 0 \le n \le 5$$

(b) This procedure performs the autocorrelation of a real sequence. Using the properties of the DFT, an alternative method may be achieved with convolution:

$$y[n] = \text{IDFT}\{X^2[k]\} = x[n] * x[n]$$

The IDFT and DFT suggest that the convolution is circular. Hence, to ensure there is no aliasing, the size of the DFT must be $N \ge 2M - 1$ where M is the length of $x[n]$. Since $M = 3, N \ge 5$.

8.13. (a)

$$g_1[n] = x[N-1-n], \qquad 0 \le n \le (N-1)$$

$$G_1[k] = \sum_{n=0}^{N-1} x[N-1-n]W_N^{kn}, \qquad 0 \le k \le (N-1)$$

Let $m = N - 1 - n$,

$$\begin{aligned} G_1[k] &= \sum_{m=0}^{N-1} x[m]W_N^{k(N-1-m)} \\ &= W_N^{k(N-1)} \sum_{m=0}^{N-1} x[m]W_N^{-km} \end{aligned}$$

Using $W_N^k = e^{-j(2\pi k/N)}, W_N^{k(N-1)} = W_N^{-k} = e^{j(2\pi k/N)}$

$$\begin{aligned} G_1[k] &= e^{j(2\pi k/N)} \sum_{m=0}^{N-1} x[m]e^{j(2\pi km/N)} \\ &= e^{j(2\pi k/N} X(e^{j\omega})|_{\omega=(2\pi k/N)} \\ G_1[k] &= H_7[k] \end{aligned}$$

(b)

$$g_2[n] = (-1)^n x[n], \qquad 0 \le n \le (N-1)$$

$$\begin{aligned}
G_2[k] &= \sum_{n=0}^{N-1} (-1)^n x[n] W_N^{kn}, \qquad 0 \le k \le (N-1) \\
&= \sum_{n=0}^{N-1} x[n] W_N^{(\frac{N}{2})n} W_N^{kn} \\
&= \sum_{n=0}^{N-1} x[n] W_N^{(k+\frac{N}{2})n} \\
&= X(e^{j\omega})|_{\omega=2\pi(k+\frac{N}{2})/N} \\
G_2[k] &= H_8[k]
\end{aligned}$$

(c)

$$g_3[n] = \begin{cases} x[n], & 0 \le n \le (N-1) \\ x[n-N], & N \le n \le (2N-1) \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
G_3[k] &= \sum_{n=0}^{2N-1} x[n] W_{2N}^{kn}, \qquad 0 \le k \le (N-1) \\
&= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} + \sum_{n=N}^{2N-1} x[n-N] W_{2N}^{kn} \\
&= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} + \sum_{m=0}^{N-1} x[m] W_{2N}^{k(m+N)} \\
&= \sum_{n=0}^{N-1} x[n] \left(1 + W_{2N}^{kN}\right) W_{2N}^{kn} \\
&= \left(1 + W_N^{(kN/2)}\right) \sum_{n=0}^{N-1} x[n] W_N^{(kn/2)} \\
&= \left(1 + (-1)^k\right) X(e^{j\omega})|_{\omega=(\pi k/N)} \\
G_3[k] &= H_3[k]
\end{aligned}$$

(d)

$$g_4[n] = \begin{cases} x[n] + x[n+N/2], & 0 \le n \le (N/2-1) \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
G_4[k] &= \sum_{n=0}^{N/2-1} \left(x[n] + x[n+\frac{N}{2}]\right) W_{N/2}^{kn}, \qquad 0 \le k \le (N-1) \\
&= \sum_{n=0}^{N/2-1} x[n] W_{N/2}^{kn} + \sum_{n=0}^{N/2-1} x[n+N/2] W_{N/2}^{kn} \\
&= \sum_{n=0}^{N/2-1} x[n] W_{N/2}^{kn} + \sum_{m=N/2}^{N-1} x[m] W_{N/2}^{k(m-N/2)} \\
&= \sum_{n=0}^{N-1} x[n] W_N^{2kn}
\end{aligned}$$

$$\begin{aligned}
&= X(e^{j\omega})|_{\omega=(4\pi k/N)} \\
G_4[k] &= H_6[k]
\end{aligned}$$

(e)

$$g_5[n] = \begin{cases} x[n], & 0 \leq n \leq (N-1) \\ 0, & N \leq n \leq (2N-1) \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
G_5[k] &= \sum_{n=0}^{2N-1} x[n] W_{2N}^{kn}, \qquad 0 \leq k \leq (N-1) \\
&= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} \\
&= X(e^{j\omega})|_{\omega=(\pi k/N)} \\
G_5[k] &= H_2[k]
\end{aligned}$$

(f)

$$g_6[n] = \begin{cases} x[n/2], & n \text{ even}, \quad 0 \leq n \leq (2N-1) \\ 0, & n \text{ odd} \end{cases}$$

$$\begin{aligned}
G_6[k] &= \sum_{n=0}^{2N-1} x[n/2] W_{2N}^{kn}, \qquad 0 \leq k \leq (N-1) \\
&= \sum_{n=0}^{N-1} x[n] W_{N}^{kn} \\
&= X(e^{j\omega})|_{\omega=(2\pi k/N)} \\
G_6[k] &= H_1[k]
\end{aligned}$$

(g)

$$g_7[n] = x[2n], \qquad 0 \leq n \leq (N/2-1)$$

$$\begin{aligned}
G_7[k] &= \sum_{n=0}^{\frac{N}{2}-1} x[2n] W_{N/2}^{kn}, \qquad 0 \leq k \leq (N-1) \\
&= \sum_{n=0}^{N-1} x[n] \left(\frac{1+(-1)^n}{2}\right) W_N^{kn} \\
&= \sum_{n=0}^{N-1} x[n] \left(\frac{1+W_N^{(N/2)n}}{2}\right) W_N^{kn} \\
&= \frac{1}{2} \sum_{n=0}^{N-1} x[n] \left(W_N^{nk} + W_N^{n(k+N/2)}\right) \\
&= \frac{1}{2} \left[X(e^{j(2\pi/N)}) + X(e^{j(2\pi/N)(k+N/2)})\right] \\
G_7[k] &= H_5[k]
\end{aligned}$$

8.14. From Table 8.2, the N-pt DFT of an N-pt sequence will be real-valued if

$$x[n] = x[((-n))_N].$$

For $0 \leq n \leq (N-1)$, this may be stated as,

$$x[n] = x[N-n], \qquad 0 \leq n \leq (N-1)$$

For this case, N = 10, and

$$\begin{aligned} x[1] &= x[9] \\ x[2] &= x[8] \\ &\vdots \end{aligned}$$

The Fourier transform of $x[n]$ displays generalized linear phase (see Section 5.7.2). This implies that for $x[n] \neq 0$, $0 \leq n \leq (N-1)$:

$$x[n] = x[N-1-n]$$

For $N = 10$,

$$\begin{aligned} x[0] &= x[9] \\ x[1] &= x[8] \\ x[2] &= x[7] \\ &\vdots \end{aligned}$$

To satify both conditions, $x[n]$ must be a constant for $0 \leq n \leq 9$.

8.15. We have two 100-pt sequences which are nonzero for the interval $0 \leq n \leq 99$.

If $x_1[n]$ is nonzero for $10 \leq n \leq 39$ only, the linear convolution

$$x_1[n] * x_2[n]$$

is a sequence of length $40 + 100 - 1 = 139$, which is nonzero for the range $10 \leq n \leq 139$.

A 100-pt circular convolution is equivalent to the linear convolution with the first 40 points aliased by the values in the range $100 \leq n \leq 139$.

Therefore, the 100-pt circular convolution will be equivalent to the linear convolution only in the range $40 \leq n \leq 99$.

8.16. (a) Since $x[n]$ is 50 points long, and $h[n]$ is 10 points long, the linear convolution $y[n] = x[n] * h[n]$ must be $50 + 10 - 1 = 59$ pts long.

(b) Circular convolution = linear convolutin + aliasing.
If we let $y[n] = x[n] * h[n]$, a more mathematical statement of the above is given by

$$x[n] \, N \, h[n] = \sum_{r=-\infty}^{\infty} y[n + rN], \qquad 0 \leq n \leq (N-1)$$

For $N = 50$,

$$x[n] \, 50 \, h[n] = y[n] + y[n + 50], \qquad 0 \leq n \leq 49$$

We are given: $x[n] \; 50 \; h[n] = 10$
Hence,

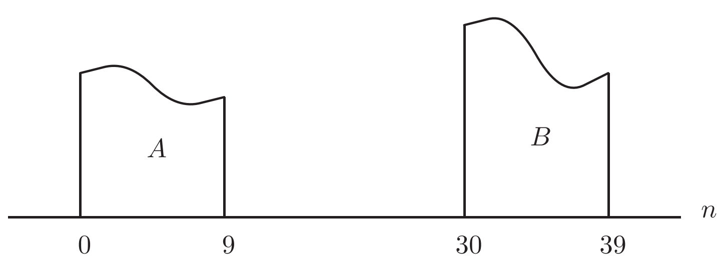
$$y[n] + y[n+50] = 10, \qquad 0 \le n \le 49$$

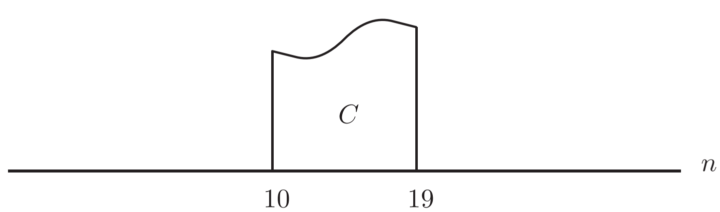
Also, $y[n] = 5, \qquad 0 \le n \le 4$.
Using the above information:

$$\begin{array}{lrcl}
n = 0 & y[0] + y[50] & = & 10 \\
\vdots & y[50] & = & 5 \\
n = 4 & y[4] + y[54] & = & 10 \\
 & y[54] & = & 5 \\
n = 5 & y[5] + y[55] & = & 10 \\
\vdots & y[55] & = & ? \\
n = 8 & y[8] + y[58] & = & 10 \\
 & y[58] & = & ? \\
n = 9 & y[9] \quad\quad & = & 10 \\
\vdots & & & \\
n = 49 & y[49] & = & 10
\end{array}$$

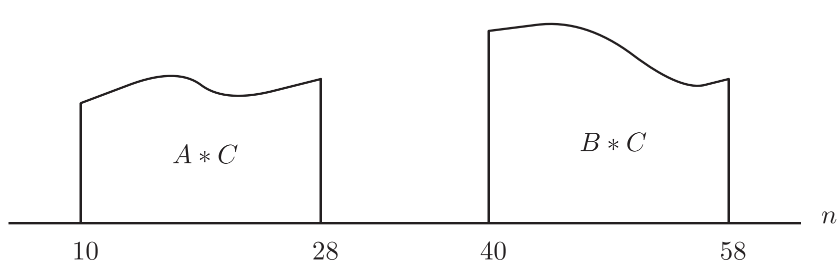
To conclude, we can determine $y[n]$ for $9 \le n \le 55$ only. (Note that $y[n]$ for $0 \le n \le 4$ is given.)

8.17. We have





(a) The linear convolution $x[n] * y[n]$ is a $40 + 20 - 1 = 59$ point sequence:



Thus, $x[n] * y[n] = w[n]$ is nonzero for $10 \leq n \leq 28$ and $40 \leq n \leq 58$.

(b) The 40-pt circluar convolution can be obtained by aliasing the linear convolution. Specifically, we alias the points in the range $40 \leq n \leq 58$ to the range $0 \leq n \leq 18$.

Since $w[n] = x[n] * y[n]$ is zero for $0 \leq n \leq 9$, the circular convolution $g[n] = x[n] \, 40 \, y[n]$ consists of only the (aliased) values:

$$w[n] = x[n] * y[n], \qquad 40 \leq n \leq 49$$

Also, the points of $g[n]$ for $18 \leq n \leq 39$ will be equivalent to the points of $w[n]$ in this range.

To conclude,

$$\begin{aligned} w[n] &= g[n], \quad 18 \leq n \leq 39 \\ w[n+40] &= g[n], \quad 0 \leq n \leq 9 \end{aligned}$$

8.18. (a) The two sequences are related by the circular shift:

$$h_2[n] = h_1[((n+4))_8]$$

Thus,

$$H_2[k] = W_8^{-4k} H_1[k]$$

and

$$|H_2[k]| = |W_8^{-4k} H_1[k]| = |H_1[k]|$$

So, yes the magnitudes of the 8-pt DFTs are equal.

(b) $h_1[n]$ is nearly like $(\sin x)/x$.

Since $H_2[k] = e^{j\pi k} H_1[k]$, $h_1[n]$ is a better lowpass filter.

8.19. (a) Overlap add:

If we divide the input into sections of length L, each section will have an output length:

$$L + 100 - 1 = L + 99$$

Thus, the required length is

$$L = 256 - 99 = 157$$

If we had 63 sections, $63 \times 157 = 9891$, there will be a remainder of 109 points. Hence, we must pad the remaining data to 256 and use another DFT.

Therefore, we require 64 DFTs and 64 IDFTs. Since $h[n]$ also requires a DFT, the total:

65 DFTs and 64 IDFTs

(b) Overlap save:

We require 99 zeros to be padded in from of the sequence. The first 99 points of the output of each section will be discarded. Thus the length after padding is 10099 points. The length of each section overlap is $256 - 99 = 157 = L$.

We require $65 \times 157 = 10205$ to get all 10099 points. Because $h[n]$ also requires a DFT:

66 DFTs and 65 IDFTs

(c) Ignoring the transients at the beginning and end of the direct convolution, each output point requires 100 multiplies and 99 adds.

overlap add:

$$\begin{aligned} \#\text{ mult} &= 129(1024) = 132096 \\ \#\text{ add} &= 129(2048) = 264192 \end{aligned}$$

overlap save:

$$\begin{aligned}
\#\ \text{mult} &= 131(1024) &= 134144 \\
\#\ \text{add} &= 131(2048) &= 268288
\end{aligned}$$

direct convolution:

$$\begin{aligned}
\#\ \text{mult} &= 100(10000) &= 1000000 \\
\#\ \text{add} &= 99(10000) &= 990000
\end{aligned}$$

8.20. First we need to compute the values $Q[0]$ and $Q[3]$:

$$\begin{aligned}
Q[0] &= X_1(1) = X_1(e^{jw})|_{w=0} \\
&= \sum_{n=0}^{\infty} x_1[n] = \frac{1}{1-\frac{1}{4}} \\
&= \frac{4}{3} \\
Q[3] &= X_1(-1) = X_1(e^{jw})|_{w=\pi} \\
&= \sum_{n=0}^{\infty} x_1[n](-1)^n \\
&= \frac{4}{3}
\end{aligned}$$

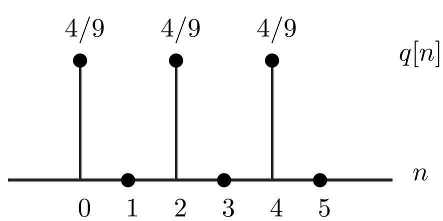
One possibility for $Q[k]$, the six-point DFT, is:

$$Q[k] = \frac{4}{3}\delta[k] + \frac{4}{3}\delta[k-3].$$

We then find $q[n]$, for $0 \le n < 6$:

$$\begin{aligned}
q[n] &= \frac{1}{6}\sum_{k=0}^{5} Q[k]e^{\frac{2\pi}{6}kn} \\
&= \frac{1}{6}(\frac{4}{3} + \frac{4}{3}(-1)^n) \\
&= \frac{2}{9}(1 + (-1)^n)
\end{aligned}$$

otherwise it's 0. Here's a sketch of q[n]:



8.21. We have:

$$DFT_7\{x_2[n]\} = X_2[k] = \sum_0^4 x_2[n]e^{-j\frac{2\pi}{7}kn}.$$

Then:

$$\begin{aligned}
x_2[0] &= \frac{1}{7}\sum_{k=0}^{6} X_2[k] \\
&= \frac{1}{7}\sum_{k=0}^{6}(Re\{X_2[k]\} + jIm\{X_2[k]\}) \\
&= \frac{1}{7}\sum_{k=0}^{6} Re\{X_2[k]\} \text{ , since } x_2[0] \text{ is real.} \\
&= g[0].
\end{aligned}$$

To determine the relationship between $x_2[1]$ and g[1], we first note that since $x_2[n]$ is real:

$$X(e^{jw}) = X^*(e^{-jw}).$$

Therefore:

$$X[k] = X^*[N-k] \text{ , } k = 0, ..., 6.$$

We thus have:

$$\begin{aligned}
g[1] &= \frac{1}{7}\sum_{k=0}^{6} Re\{X_2[k]\}W_7^{-k} \\
&= \frac{1}{7}\sum_{k=0}^{6} \frac{X_2[k] + X_2^*[k]}{2}W_7^{-k} \\
&= \frac{1}{7}\sum_{k=0}^{6} \frac{X_2[k]}{2}W_7^{-k} + \frac{1}{7}\sum_{k=0}^{6} \frac{X_2[N-k]}{2}W_7^{-k} \\
&= \frac{1}{2}x_2[1] + \frac{1}{14}\sum_{k=0}^{6} X_2[k]W_7^{k} \\
&= \frac{1}{2}x_2[1] + \frac{1}{14}\sum_{k=0}^{6} X_2[k]W_7^{-6k} \\
&= \frac{1}{2}(x_2[1] + x_2[6]) \\
&= \frac{1}{2}(x_2[1] + 0) \\
&= \frac{1}{2}x_2[1].
\end{aligned}$$

8.22. (i) This corresponds to $x_i[n] = x_i^*[((-n))_N]$, where $N = 5$. Note that this is only true for $x_2[n]$.

(ii) $X_i(e^{jw})$ has linear phase corresponds to $x_i[n]$ having some internal symmetry, this is only true for $x_1[n]$.

(iii) The DFT has linear phase corresponds to $\tilde{x}_i[n]$ (the periodic sequence obtained from $x_i[n]$) being symmetric, this is true for $x_1[n]$ and $x_2[n]$ only.

8.23. (a) No. x[n] only has N degrees of freedom and we have $M \geq N$ constraints which can only be satisfied if $x[n] = 0$. Specifically, we want

$$X(e^{j\frac{2\pi k}{M}}) = DFT_M\{x[n]\} = 0.$$

Since $M \geq N$, there is no aliasing and x[n] can be expressed as:

$$x[n] = \frac{1}{M}\sum_{k=0}^{M-1} X[k]W_M^{kn} \ , \ n = 0, ..., M-1.$$

Where $X[k]$ is the M-point DFT of $x[n]$, since $X[k] = 0$, we thus conclude that $x[n] = 0$, and therefore the answer is NO.

(b) Here, we only need to make sure that when time-aliased to M samples, $x[n]$ is all zeros. For example, let

$$x[n] = \delta[n] - \delta[n-2]$$

then,

$$X(e^{jw}) = 1 - e^{-2jw}.$$

Let $M = 2$, then we have

$$\begin{aligned} X(e^{j\frac{2\pi}{2}0}) &= 1 - 1 = 0 \\ X(e^{j\frac{2\pi}{2}1}) &= 1 - 1 = 0 \end{aligned}$$

8.24. $x_2[n]$ is $x_1[n]$ time aliased to have only N samples. Since

$$x_1[n] = (\frac{1}{3})^n u[n],$$

We get:

$$x_2[n] = \begin{cases} \frac{(\frac{1}{3})^n}{1-(\frac{1}{3})^N} & , \quad n = 0, ..., N-1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Solutions – Chapter 9

Computation of the Discrete-Time Fourier Transform

9.1.

$$
\begin{aligned}
Y[k] &= \sum_{n=0}^{2N-1} y[n]e^{-j(\frac{2\pi}{2N})kn} \\
&= \sum_{n=0}^{N-1} e^{-j(\pi/N)n^2} e^{-j(2\pi/N)(k/2)n} + \sum_{n=N}^{2N-1} e^{-j(\pi/N)n^2} e^{-j(2\pi/N)(k/2)n} \\
&= \sum_{n=0}^{N-1} e^{-j(\pi/N)n^2} e^{-j(2\pi/N)(k/2)n} + \sum_{l=0}^{N-1} e^{-j(\pi/N)(l+N)^2} e^{-j(2\pi/N)(k/2)(l+N)} \\
&= \sum_{n=0}^{N-1} e^{-j(\pi/N)n^2} e^{-j(2\pi/N)(k/2)n} + e^{-j\pi k} \sum_{l=0}^{N-1} e^{-j(\pi/N)(l^2+2Nl+N^2)} e^{-j(2\pi/N)(k/2)l} \\
&= \sum_{n=0}^{N-1} e^{-j(\pi/N)n^2} e^{-j(2\pi/N)(k/2)n} + (-1)^k \sum_{l=0}^{N-1} e^{-j(\pi/N)l^2} e^{-j(2\pi/N)(k/2)l} \\
&= (1+(-1)^k) \sum_{n=0}^{N-1} e^{-j(\pi/N)n^2} e^{-j(2\pi/N)(k/2)n} \\
&= \begin{cases} 2X[k/2], & k \text{ even} \\ 0, & k \text{ odd} \end{cases}
\end{aligned}
$$

Thus,

$$
Y[k] = \begin{cases} 2\sqrt{N}e^{-j\pi/4}e^{j(\pi/N)k^2/4}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}
$$

9.2. (a) We offer two solutions to this problem.

Solution #1: Looking at the DFT of the sequence, we find

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{(N/2)-1} x[n]e^{-j2\pi kn/N} + \sum_{n=N/2}^{N-1} x[n]e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{(N/2)-1} x[n]e^{-j2\pi kn/N} + \sum_{r=0}^{(N/2)-1} x[r+(N/2)]e^{-j2\pi k[r+(N/2)]/N} \\
&= \sum_{n=0}^{(N/2)-1} x[n][1-(-1)^k]e^{-j2\pi kn/N} \\
&= 0, \quad k \text{ even}
\end{aligned}
$$

Solution #2: Alternatively, we can use the circular shift property of the DFT to find

$$
\begin{aligned}
X[k] &= -X[k]e^{-j(\frac{2\pi}{N})k(\frac{N}{2})} \\
&= -(-1)^k X[k] \\
&= (-1)^{k+1} X[k]
\end{aligned}
$$

When k is even, we have $X[k] = -X[k]$ which can only be true if $X[k] = 0$.

(b) Evaluating the DFT at the odd-indexed samples gives us

$$
\begin{aligned}
X[2k+1] &= \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)(2k+1)n} \\
&= \sum_{n=0}^{N/2-1} x[n]e^{-j2\pi n/N}e^{-j2\pi kn/(N/2)} + \sum_{n=N/2}^{N-1} x[n]e^{-j2\pi n/N}e^{-j2\pi kn/(N/2)} \\
&= \mathrm{DFT}_{N/2}\left\{x[n]e^{-j(2\pi/N)n}\right\} + \sum_{l=0}^{N/2-1} x[l+(N/2)]e^{-j2\pi[l+(N/2)]/N}e^{-j2\pi k[l+(N/2)]/(N/2)} \\
&= \mathrm{DFT}_{N/2}\left\{x[n]e^{-j(2\pi/N)n}\right\} + (-1)(-1)\sum_{l=0}^{N/2-1} x[l]e^{-j2\pi l/N}e^{-j2\pi kl/(N/2)} \\
&= \mathrm{DFT}_{N/2}\left\{2x[n]e^{-j(2\pi/N)n}\right\}
\end{aligned}
$$

for $k = 0, \ldots, N/2-1$. Thus, we can compute the odd-indexed DFT values using one $N/2$ point DFT plus a small amount of extra computation.

Solutions – Chapter 10

Fourier Analysis of Signals

Using the Discrete Fourier Transform

10.1. (a) Starting with definition of the time-dependent Fourier transform,

$$Y[n,\lambda) = \sum_{m=-\infty}^{\infty} y[n+m]w[m]e^{-j\lambda m}$$

we plug in

$$y[n+m] = \sum_{k=0}^{M} h[k]x[n+m-k]$$

to get

$$\begin{aligned} Y[n,\lambda) &= \sum_{m=-\infty}^{\infty}\sum_{k=0}^{M} h[k]x[n+m-k]w[m]e^{-j\lambda m} \\ &= \sum_{k=0}^{M} h[k] \sum_{m=-\infty}^{\infty} x[n+m-k]w[m]e^{-j\lambda m} \\ &= \sum_{k=0}^{M} h[k]X[n-k,\lambda) \\ &= h[n] * X[n,\lambda) \end{aligned}$$

where the convolution is for the variable n.

(b) Starting with

$$\check{Y}[n,\lambda) = e^{-j\lambda n}Y[n,\lambda)$$

we find

$$\begin{aligned} \check{Y}[n,\lambda) &= e^{-j\lambda n}\left[\sum_{k=0}^{M} h[k]X[n-k,\lambda)\right] \\ &= e^{-j\lambda n}\left[\sum_{k=0}^{M} h[k]e^{j(n-k)\lambda}\check{X}[n-k,\lambda)\right] \\ &= \sum_{k=0}^{M} h[k]e^{-j\lambda k}\check{X}[n-k,\lambda) \end{aligned}$$

If the window is long compared to M, then a small time shift in $\check{X}[n,\lambda)$ won't radically alter the spectrum, and

$$\check{X}[n-k,\lambda) \simeq \check{X}[n,\lambda)$$

Consequently,

$$\begin{aligned} \check{Y}[n,\lambda) &\simeq \sum_{k=0}^{M} h[k]e^{-j\lambda k}\check{X}[n,\lambda) \\ &\simeq H(e^{j\lambda})\check{X}[n,\lambda) \end{aligned}$$