Chapter 2

Discrete-Time Signals and Systems

Problems

Basic Problems

2.1. For each of the pairs of sequences in Figure P2.1-1, use discrete convolution to find the response to the input $x[n]$ of the linear time-invariant system with impulse response $h[n]$.

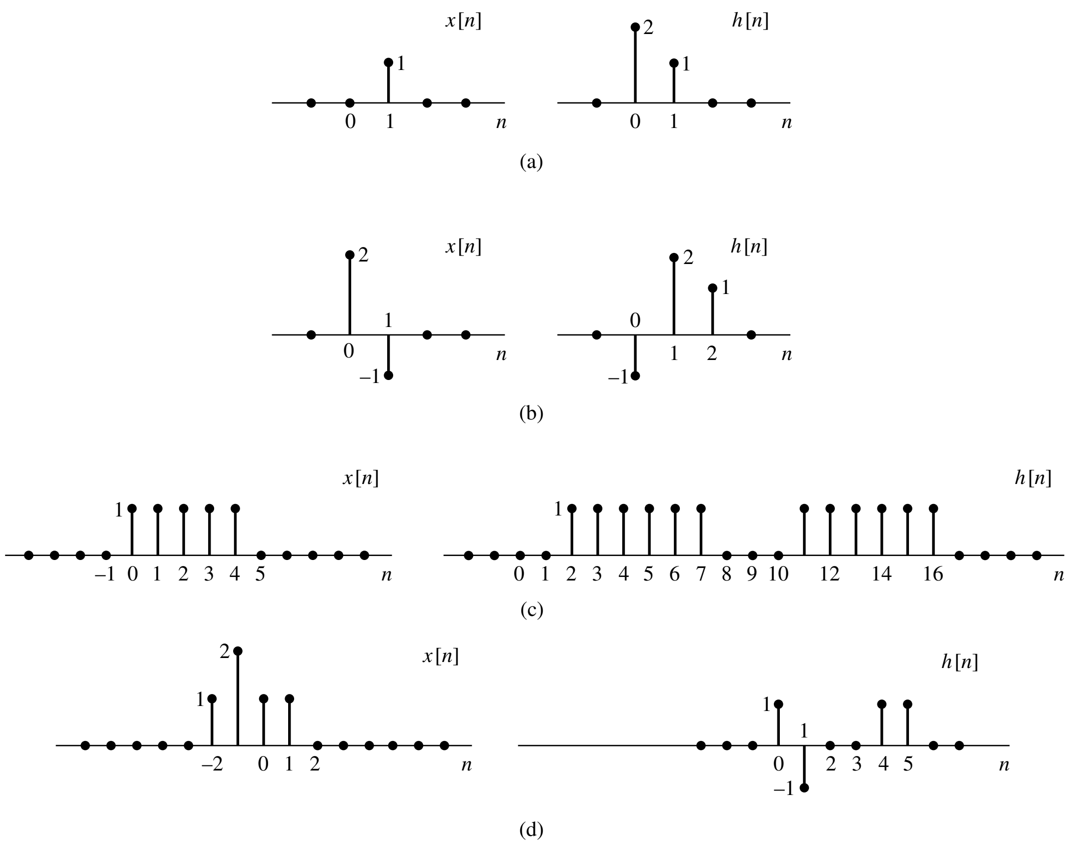


Figure P2.1-1

2.2. The impulse response of a linear time-invariant system is shown in Figure P2.2-1. Determine and carefully sketch the response of this system to the input $x[n] = u[n-4]$.

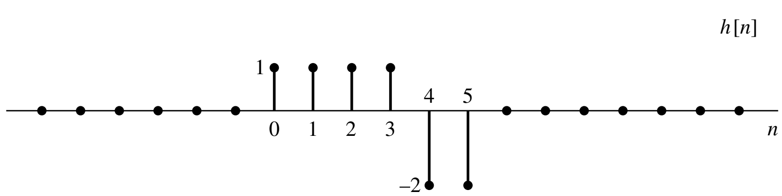


Figure P2.2-1

2.3. A linear time-invariant system has impulse response $h[n] = u[n]$. Determine the response of this system to the input $x[n]$ shown in Figure P2.3-1 and described as

$$x[n] = \begin{cases} 0, & n < 0, \\ a^n, & 0 \le n \le N_1, \\ 0, & N_1 < n < N_2, \\ a^{n-N_2}, & N_2 \le n \le N_2 + N_1, \\ 0, & N_2 + N_1 < n, \end{cases}$$

where $0 < a < 1$.

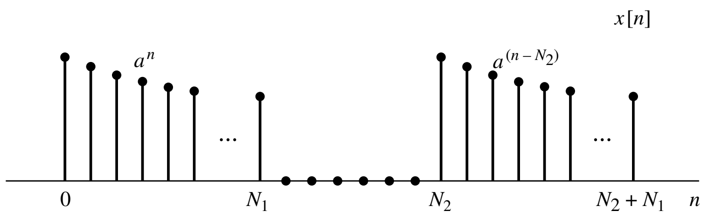


Figure P2.3-1

2.4. Which of the following discrete-time signals could be eigenfunctions of any stable LTI system?

(a) $5^n u[n]$

(b) $e^{j2\omega n}$

(c) $e^{j\omega n} + e^{j2\omega n}$

(d) 5^n

(e) $5^n \cdot e^{j2\omega n}$

2.5. Three systems A, B, and C have the inputs and outputs indicated in Figure P2.5-1. Determine whether each system could be LTI. If your answer is yes, specify whether there could be more than one LTI system with the given input–output pair. Explain your answer.

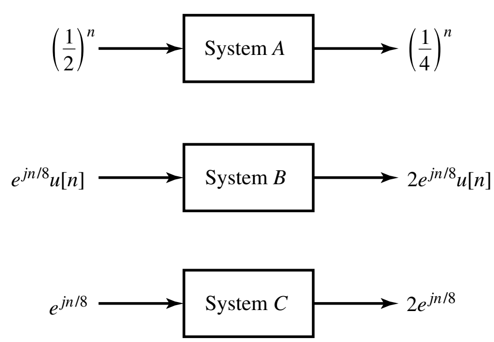


Figure P2.5-1

2.6. Consider the difference equation

$$y[n] + \frac{1}{15}y[n-1] - \frac{2}{15}y[n-2] = x[n].$$

(a) Determine the general form of the homogeneous solution to this equation.

(b) Both a causal and an anticausal LTI system are characterized by the given difference equation. Find the impulse responses of the two systems.

(c) Show that the causal LTI system is stable and the anticausal LTI system is unstable.

(d) Find a particular solution to the difference equation when $x[n] = (3/5)^n\, u[n]$.

2.7. Consider an LTI system with frequency response

$$H(e^{j\omega}) = e^{-j(\omega - \frac{\pi}{4})}\left(\frac{1 + e^{-j2\omega} + 4e^{-j4\omega}}{1 + \frac{1}{2}e^{-j2\omega}}\right), \qquad -\pi < \omega \le \pi.$$

Determine the output $y[n]$ for all n if the input for all n is

$$x[n] = \cos\left(\frac{\pi n}{2}\right).$$

2.8. The input–output pair shown in Figure P2.8-1 is given for a stable LTI system.

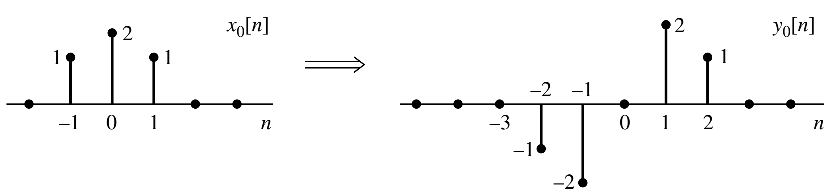


Figure P2.8-1

(a) Determine the response to the input $x_1[n]$ in Figure P2.8-2.

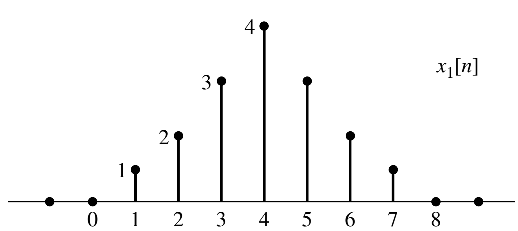


Figure P2.8-2

(b) Determine the impulse response of the system.

Advanced Problems

2.9. A linear time-invariant system has impulse response $h[n] = a^n u[n]$.

(a) Determine $y_1[n]$, the response of the system to the input $x_1[n] = e^{j(\pi/2)n}$.

(b) Use the result of Part (a) to help to determine $y_2[n]$, the response of the system to the input $x_2[n] = \cos(\pi n/2)$.

(c) Determine $y_3[n]$, the response of the system to the input $x_3[n] = e^{j(\pi/2)n}u[n]$.

(d) Compare $y_3[n]$ with $y_1[n]$ for large n.

2.10. The frequency response of an LTI system is

$$H(e^{j\omega}) = e^{-j\omega/4}, \qquad -\pi < \omega \le \pi.$$

Determine the output of the system, $y[n]$, when the input is $x[n] = \cos(5\pi n/2)$. Express your answer in as simple a form as you can.

2.11. Consider the cascade of LTI discrete-time systems shown in Figure P2.11-1.

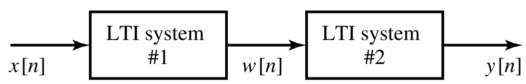


Figure P2.11-1

The first system is described by the equation

$$H_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < 0.5\pi, \\ 0, & 0.5\pi \le |\omega| < \pi, \end{cases}$$

and the second system is described by the equation

$$y[n] = w[n] - w[n-1].$$

The input to this system is

$$x[n] = \cos(0.6\pi n) + 3\delta[n-5] + 2.$$

Determine the output $y[n]$. With careful thought, you will be able to use the properties of LTI systems to write down the answer by inspection.

2.12. For the system shown in Figure P2.12-1, System 1 is a memoryless nonlinear system. System 2 determines the value of A according to the relation

$$A = \sum_{n=0}^{100} y[n].$$

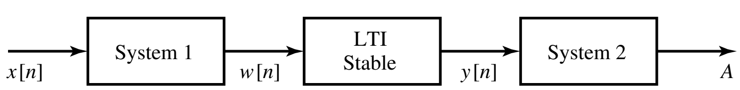


Figure P2.12-1

Specifically, consider the class of inputs of the form $x[n] = \cos(\omega n)$, with ω a real finite number. Varying the value of ω at the input will change A; i.e., A will be a function of ω. In general, will A be periodic in ω? Justify your answer.

Chapter 3

The z-transform

Problems

Basic Problems with Answers

3.1. An LTI system is characterized by the system function

$$H(z) = \frac{\left(1 - \frac{1}{2}z^{-2}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}, \qquad |z| > \frac{1}{2}.$$

(a) Determine the impulse response of the system.

(b) Determine the difference equation relating the system input $x[n]$ and the system output $y[n]$.

3.2. Consider an LTI system that is stable and for which $H(z)$, the z-transform of the impulse response, is given by

$$H(z) = \frac{3 - 7z^{-1} + 5z^{-2}}{1 - \frac{5}{2}z^{-1} + z^{-2}}.$$

Suppose $x[n]$, the input to the system, is a unit step sequence.

(a) Find the output $y[n]$ by evaluating the discrete convolution of $x[n]$ and $h[n]$.

(b) Find the output $y[n]$ by computing the inverse z-transform of $Y(z)$.

3.3. Consider a stable linear time-invariant system. The z-transform of the impulse response is

$$H(z) = \frac{z^{-1} + z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)}.$$

Suppose $x[n]$, the input to the system, is $2u[n]$. Determine $y[n]$ at $n = 1$.

3.4. Suppose the z-transform of $x[n]$ is

$$X(z) = \frac{z^{10}}{\left(z-\frac{1}{2}\right)\left(z-\frac{3}{2}\right)^{10}\left(z+\frac{3}{2}\right)^{2}\left(z+\frac{5}{2}\right)\left(z+\frac{7}{2}\right)}.$$

It is also known that $x[n]$ is a stable sequence.

(a) Determine the region of convergence of $X(z)$.

(b) Determine $x[n]$ at $n = -8$.

3.5. When the input to a causal LTI system is

$$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^{n} u[n] - \frac{4}{3}2^{n}u[-n-1],$$

the z-transform of the output is

$$Y(z) = \frac{1+z^{-1}}{\left(1-z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}.$$

(a) Find the z-transform of $x[n]$.

(b) What is the region of convergence of $Y(z)$?

(c) Find the impulse response of the system.

(d) Is the system stable?

Chapter 4

Sampling of continuous-time signals

Problems

Basic Problems

4.1. A complex-valued continuous-time signal $x_c(t)$ has the Fourier transform shown in Figure P4.1-1, where $(\Omega_2 - \Omega_1) = \Delta\Omega$. This signal is sampled to produce the sequence $x[n] = x_c(nT)$.

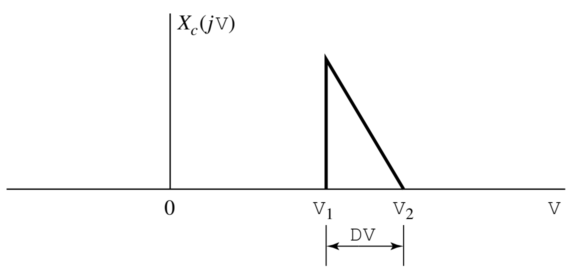


Figure P4.1-1

(a) Sketch the Fourier transform $X(e^{j\omega})$ of the sequence $x[n]$ for $T = \pi/\Omega_2$.

(b) What is the *lowest* sampling frequency that can be used without incurring any aliasing distortion, i.e., so that $x_c(t)$ can be recovered from $x[n]$?

(c) Draw the block diagram of a system that can be used to recover $x_c(t)$ from $x[n]$ if the sampling rate is greater than or equal to the rate determined in Part (b). Assume that (complex) ideal filters are available.

4.2. A continuous-time signal $x_c(t)$, with Fourier transform $X_c(j\Omega)$ shown in Figure P4.2-1, is sampled with sampling period $T = 2\pi/\Omega_0$ to form the sequence $x[n] = x_c(nT)$.

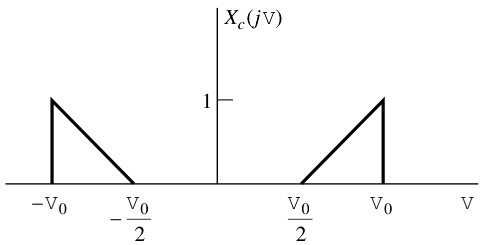


Figure P4.2-1

(a) Sketch the Fourier transform $X(e^{j\omega})$ for $|\omega| < \pi$.

(b) The signal $x[n]$ is to be transmitted across a digital channel. At the receiver, the original signal $x_c(t)$ must be recovered. Draw a block diagram of the recovery system and specify its characteristics. Assume that ideal filters are available.

(c) In terms of Ω_0, for what range of values of T can $x_c(t)$ be recovered from $x[n]$?

4.3. Consider the sequence $x[n]$ whose Fourier transform $X(e^{j\omega})$ is shown in Figure P4.3-1. Define

$$x_s[n] = \begin{cases} x[n], & n = Mk, \quad k = 0, \pm 1, \pm 2, \ldots, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$x_d[n] = x_s[Mn] = x[Mn].$$

(a) Sketch $X_s(e^{j\omega})$ and $X_d(e^{j\omega})$ for each of the following cases:

(i) $M = 3$, $\omega_H = \pi/2$

(ii) $M = 3$, $\omega_H = \pi/4$

(b) What is the maximum value of ω_H that will avoid aliasing when $M = 3$?

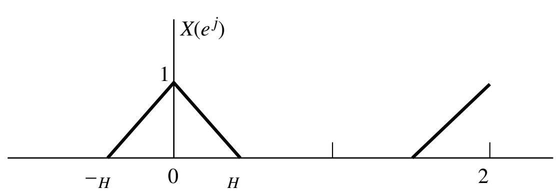


Figure P4.3-1

4.4. Using Parseval's theorem, briefly explain why the amplitude of the Fourier transform changes during downsampling but not during upsampling.

4.5. **(a)** Is the system in DTSP3E Figure 4.9 linear for a given choice of T? If so, provide a brief argument demonstrating that it satisfies linearity. If not, provide a counterexample.

(b) Is the system in DTSP3E Figure 4.9 time invariant for a given choice of T? If so, provide a brief argument demonstrating that it satisfies time invariance. If not, provide a counterexample.

Advanced Problems

4.6. Consider the systems shown in Figure P4.6-1. Suppose that $H_1(e^{j\omega})$ is fixed and known. Find $H_2(e^{j\omega})$, the frequency response of an LTI system, such that $y_2[n] = y_1[n]$ if the inputs to the systems are the same.

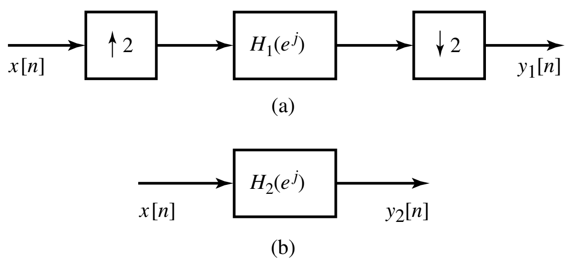


Figure P4.6-1

4.7. In the system of DTSP3E Figure 4.9, assume that the discrete-time system is linear and time invariant and that $X_c(j\Omega) = 0$ for $|\Omega| \geq 4000\pi$. Determine the largest possible value for T and the corresponding frequency response $H(e^{j\omega})$ for the discrete-time system such that

$$Y_c(j\Omega) = \begin{cases} |\Omega| X_c(j\Omega), & 1000\pi < |\Omega| < 2000\pi, \\ 0, & \text{otherwise.} \end{cases}$$

4.8. In the system of DTSP3E Figure 4.9, assume that $X_c(j\Omega) = 0$ for $|\Omega| > \pi/T$. Determine and plot the magnitude and phase of the frequency response of the discrete-time LTI system such that the output $y_r(t)$ is the running integral of the input, i.e.,

$$y_r(t) = \int_{-\infty}^{t} x_c(\tau) d\tau.$$

4.9. A bandlimited continuous-time signal is known to contain a 60-Hz component, which we want to remove by processing with the system of DTSP3E Figure 4.9, where $T = 10^{-4}$.

(a) What is the highest frequency that the continuous-time signal can contain if aliasing is to be avoided?

(b) The discrete-time system to be used has frequency response

$$H(e^{j\omega}) = \frac{[1 - e^{-j(\omega-\omega_0)}][1 - e^{-j(\omega+\omega_0)}]}{[1 - 0.9e^{-j(\omega-\omega_0)}][1 - 0.9e^{-j(\omega+\omega_0)}]}.$$

Sketch the magnitude and phase of $H(e^{j\omega})$.

(c) What value should be chosen for ω_0 to eliminate the 60-Hz component?

4.10. For the LTI system in Figure P4.10-1,

$$H(e^{j\omega}) = e^{-j\omega/2}, \qquad |\omega| \le \pi \text{ (half-sample delay)}.$$

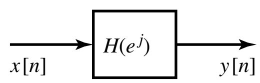


Figure P4.10-1

(a) Determine a choice for T and $h_c(t)$ in the system of DTSP3E Figure 4.14 so that the system in Figure P4.10-1 with $H(e^{j\omega})$ as specified is equivalent to the system in DTSP3E Figure 4.14.

(b) Determine and sketch $y[n]$ when the input sequence is

$$x[n] = \cos\left(\frac{5\pi}{2}n - \frac{\pi}{4}\right),$$

as sketched in Figure P4.10-2.

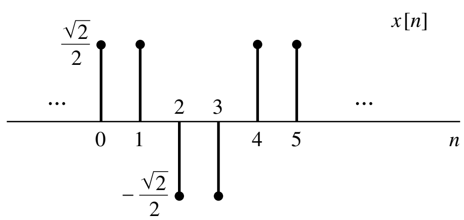


Figure P4.10-2

4.11. Consider the system of DTSP3E Figure 4.14 with the continuous-time LTI system causal and characterized by the linear constant-coefficient differential equation

$$\frac{d^2y_c(t)}{dt^2} + 4\frac{dy_c(t)}{dt} + 3y_c(t) = x_c(t).$$

The overall system is equivalent to a causal discrete-time LTI system. Determine the frequency response $H(e^{j\omega})$ of the equivalent discrete-time system when $T = 0.1$s.

4.12. In Figure P4.12-1, $x[n] = x_c(nT)$ and $y[n] = x[2n]$.

(a) Assume that $x_c(t)$ has a Fourier transform such that $X_c(j\Omega) = 0$, $|\Omega| > 2\pi(100)$. What value of T is required so that

$$X(e^{j\omega}) = 0, \qquad \frac{\pi}{2} < |\omega| \leq \pi?$$

(b) How should T_d be chosen so that $y_c(t) = x_c(t)$?

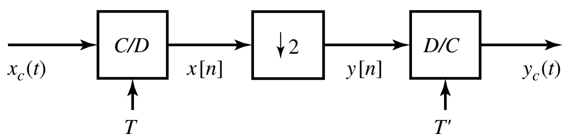


Figure P4.12-1

4.13. Suppose that you obtained a sequence $s[n]$ by filtering a speech signal $s_c(t)$ with a continuous-time lowpass filter with a cutoff frequency of 5 kHz and then sampling the resulting output at a 10-kHz rate, as shown in Figure P4.13-1. Unfortunately, the speech signal $s_c(t)$ was destroyed once the sequence $s[n]$ was stored on magnetic tape. Later, you find that what you should have done is followed the process shown in Figure P4.13-2. Develop a method to obtain $s_1[n]$ from $s[n]$ using discrete-time processing. Your method may require a very large amount of computation, but should *not* require a C/D or D/C converter. If your method uses a discrete-time filter, you should specify the frequency response of the filter.

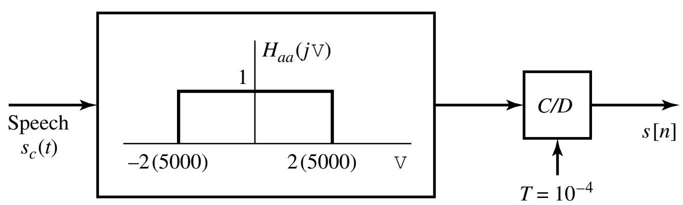


Figure P4.13-1

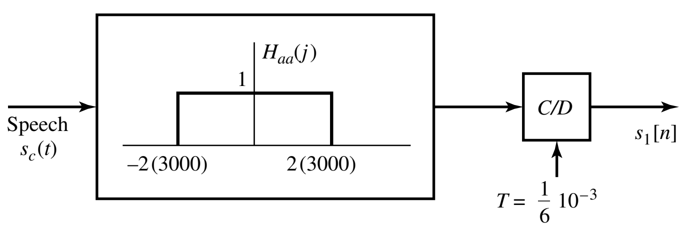


Figure P4.13-2

4.14. Consider the system shown in Figure P4.14-1, where

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \pi/L, \\ 0, & \pi/L < |\omega| \leq \pi. \end{cases}$$

Sketch $Y_c(j\Omega)$ if $X_c(j\Omega)$ is as shown in Figure P4.14-2.

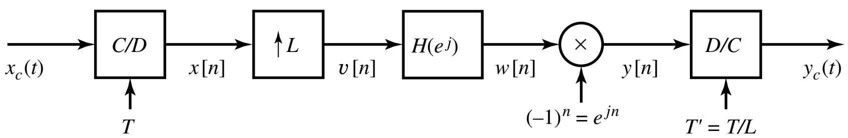


Figure P4.14-1

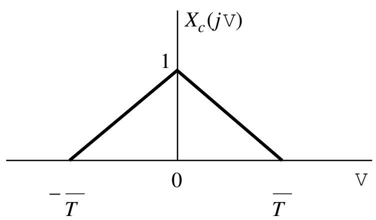


Figure P4.14-2

4.15. The system shown in Figure P4.15-1 approximately interpolates the sequence $x[n]$ by a factor L. Suppose that the linear filter has impulse response $h[n]$ such that $h[n] = h[-n]$ and $h[n] = 0$ for $|n| > (RL - 1)$, where R and L are integers; i.e., the impulse response is symmetric and of length $(2RL - 1)$ samples.

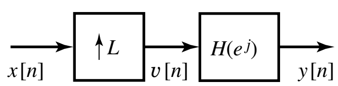


Figure P4.15-1

(a) In answering the following, do not be concerned about the causality of the system; it can be made causal by including some delay. Specifically, how much delay must be inserted to make the system causal?

(b) What conditions must be satisfied by $h[n]$ in order that $y[n] = x[n/L]$ for $n = 0, \pm L, \pm 2L, \pm 3L, \ldots$?

(c) By exploiting the symmetry of the impulse response, show that each sample of $y[n]$ can be computed with no more than RL multiplications.

(d) By taking advantage of the fact that multiplications by zero need not be done, show that only $2R$ multiplications per output sample are required.

4.16. Consider the system shown in Figure P4.16-1. The input to this system is the bandlimited signal whose Fourier transform is shown in Figure P4.16-3 with $\Omega_0 = \pi/T$. The discrete-time LTI system in Figure P4.16-1 has the frequency response shown in Figure P4.16-2.

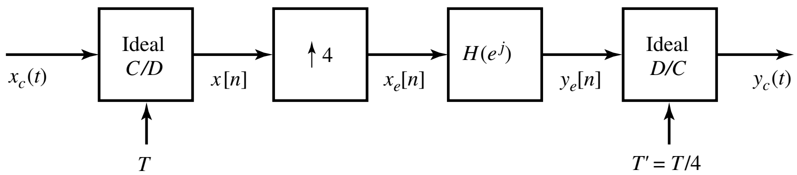


Figure P4.16-1

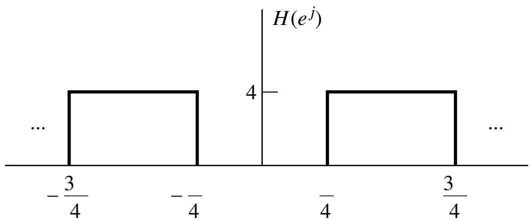


Figure P4.16-2

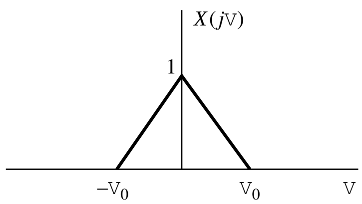


Figure P4.16-3

(a) Sketch the Fourier transforms $X(e^{j\omega})$, $X_e(e^{j\omega})$, $Y_e(e^{j\omega})$, and $Y_c(j\Omega)$.

(b) For the general case when $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$, express $Y_c(j\Omega)$ in terms of $X_c(j\Omega)$. Also, give a general expression for $y_c(t)$ in terms of $x_c(t)$ when $x_c(t)$ is bandlimited in this manner.

4.17. Let $x_c(t)$ be a real-valued continuous-time signal with highest frequency $2\pi(250)$ radians/second. Furthermore, let $y_c(t) = x_c(t - 1/1000)$.

(a) If $x[n] = x_c(n/500)$, is it theoretically possible to recover $x_c(t)$ from $x[n]$? Justify your answer.

(b) If $y[n] = y_c(n/500)$, is it theoretically possible to recover $y_c(t)$ from $y[n]$? Justify your answer.

(c) Is it possible to obtain $y[n]$ from $x[n]$ using the system in Figure P4.17-1? If so, determine $H_1(e^{j\omega})$.

(d) It is also possible to obtain $y[n]$ from $x[n]$ without any upsampling or downsampling using a single LTI system with frequency response $H_2(e^{j\omega})$. Determine $H_2(e^{j\omega})$.

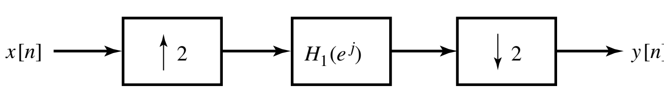


Figure P4.17-1

4.18. Consider the system shown in Figure P4.18-1 for discrete-time processing of the continuous-time input signal $g_c(t)$. The input signal $g_c(t)$ is of the form $g_c(t) = f_c(t) + e_c(t)$, where the Fourier transforms of $f_c(t)$ and $e_c(t)$ are shown in Figure P4.18-2. Since the input signal is not bandlimited, a continuous-time antialiasing filter $H_{\text{aa}}(j\Omega)$ is used. The magnitude of the frequency response for $H_{\text{aa}}(j\Omega)$ is shown in Figure P4.18-3, and the phase response of the antialiasing filter is $\angle H_{\text{aa}}(j\Omega) = -\Omega^3$.

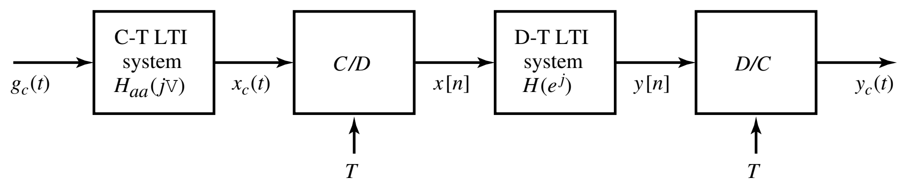


Figure P4.18-1

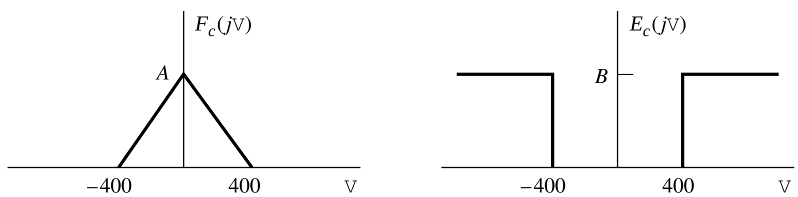


Figure P4.18-2

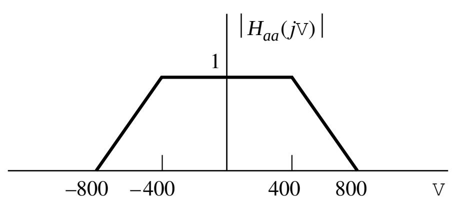


Figure P4.18-3

(a) If the sampling rate is $2\pi/T = 1600\pi$, determine the magnitude and phase of $H(e^{j\omega})$, the frequency response of the discrete-time system, so that the output is $y_c(t) = f_c(t)$.

(b) Is it possible that $y_c(t) = f_c(t)$ if $2\pi/T < 1600\pi$? If so, what is the *minimum* value of $2\pi/T$? Determine $H(e^{j\omega})$ for this choice of $2\pi/T$.

4.19. Assume that the continuous-time signal $x_c(t)$ in Figure P4.19-1 is exactly bandlimited and exactly time limited so that

$$x_c(t) = 0 \text{ for } t < 0 \text{ and } t > 10 \text{ seconds}$$

and

$$X_c(j\Omega) = 0 \text{ for } |\Omega| \geq 2\pi \times 10^4.$$

While no continuous-time signal can be exactly bandlimited and time limited, the assumption that a signal satisfies both constraints often an excellent approximation and one that we typically rely on in discrete-time processing of continuous-time signals. The continuous-time signal $x_c(t)$ is sampled as indicated in Figure P4.19-1 to obtain the sequence $x[n]$, which we want to process to estimate the total area A under $x_c(t)$ as precisely possible.

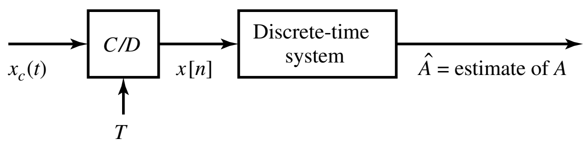


Figure P4.19-1

Specifically, we define

$$A = \int_0^{10} x_c(t)\, dt.$$

Specify a choice for the impulse response $h[n]$ for the discrete-time system and the largest possible value of T to obtain as accurate an estimate as possible of A. State specifically whether your estimate will be exact or approximate.

4.20. Consider the system in Figure P4.20-1 with $H_0(z)$, $H_1(z)$, and $H_2(z)$ as the system functions of LTI systems. Assume that $x[n]$ is an arbitrary stable complex signal without any symmetry properties.

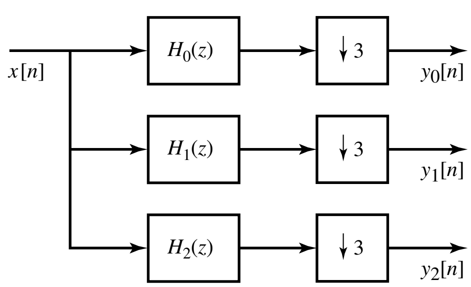


Figure P4.20-1

(a) Let $H_0(z) = 1$, $H_1(z) = z^{-1}$, and $H_2(z) = z^{-2}$. Can you reconstruct $x[n]$ from $y_0[n]$, $y_1[n]$, and $y_2[n]$? If so, how? If not, justify your answer.

(b) Assume that $H_0(e^{j\omega})$, $H_1(e^{j\omega})$, and $H_2(e^{j\omega})$ are as follows:

$$H_0(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \pi/3, \\ 0, & \text{otherwise,} \end{cases}$$
$$H_1(e^{j\omega}) = \begin{cases} 1, & \pi/3 < |\omega| \leq 2\pi/3, \\ 0, & \text{otherwise,} \end{cases}$$
$$H_2(e^{j\omega}) = \begin{cases} 1, & 2\pi/3 < |\omega| \leq \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Can you reconstruct $x[n]$ from $y_0[n]$, $y_1[n]$, and $y_2[n]$? If so, how? If not, justify your answer.

Now consider the system in Figure P4.20-2. Let $H_3(e^{j\omega})$ and $H_4(e^{j\omega})$ be the frequency responses of the LTI systems in this figure. Again, assume that $x[n]$ is an arbitrary stable complex signal with no symmetry properties.

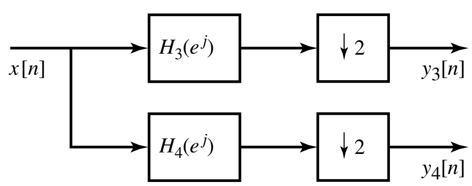


Figure P4.20-2

(c) Suppose that $H_3(e^{j\omega}) = 1$ and

$$H_4(e^{j\omega}) = \begin{cases} 1, & 0 \leq \omega < \pi, \\ -1, & -\pi \leq \omega < 0. \end{cases}$$

Can you reconstruct $x[n]$ from $y_3[n]$ and $y_4[n]$? If so, how? If not, justify your answer.

Chapter 5

Transform analysis of linear time-invariant systems

Problems

Basic Problems

5.1. Consider a causal linear time-invariant system with system function

$$H(z) = \frac{1 - a^{-1}z^{-1}}{1 - az^{-1}},$$

where a is real.

(a) Write the difference equation that relates the input and the output of this system.

(b) For what range of values of a is the system stable?

(c) For $a = \frac{1}{2}$, plot the pole–zero diagram and shade the region of convergence.

(d) Find the impulse response $h[n]$ for the system.

(e) Show that the system is an all-pass system, i.e., that the magnitude of the frequency response is a constant. Also, specify the value of the constant.

5.2. **(a)** For each of the four types of causal linear phase FIR filters discussed in DTSP3E Section 5.7.3, determine whether the associated symmetry imposes any constraint on the frequency response at $\omega = 0$ and/or $\omega = \pi$.

(b) For each of the following types of desired filter, indicate which of the four FIR filter types would be useful to consider in approximating the desired filter:

Lowpass
Bandpass
Highpass
Bandstop
Differentiator

5.3. Let $x[n]$ be a causal, N-point sequence that is zero outside the range $0 \leq n \leq N-1$. When $x[n]$ is the input to the causal LTI system represented by the difference equation

$$y[n] - \tfrac{1}{4}y[n-2] = x[n-2] - \tfrac{1}{4}x[n],$$

the output is $y[n]$, also a causal, N-point sequence.

(a) Show that the causal LTI system described by this difference equation represents an all-pass filter.

(b) Given that

$$\sum_{n=0}^{N-1} |x[n]|^2 = 5,$$

determine the value of

$$\sum_{n=0}^{N-1} |y[n]|^2.$$

5.4. Is the following statement true or false?

Statement: It is not possible for a noncausal system to have a positive constant group delay; i.e., $\text{grd}\left[H(e^{j\omega})\right] = \tau_0 > 0$.

If the statement is true, give a brief argument justifying it. If the statement is false, provide a counterexample.

5.5. An LTI system with impulse response $h_1[n]$ is an ideal lowpass filter with cutoff frequency $\omega_c = \pi/2$. The frequency response of the system is $H_1(e^{j\omega})$. Suppose a new LTI system with impulse response $h_2[n]$ is obtained from $h_1[n]$ by

$$h_2[n] = (-1)^n h_1[n].$$

Sketch the frequency response $H_2(e^{j\omega})$.

Advanced Problems

5.6. A signal $x[n]$ is processed by a linear time-invariant system $H(z)$ and then downsampled by a factor of 2 to yield $y[n]$, as shown in Figure P5.6-1. The pole–zero plot for $H(z)$ is shown in Figure P5.6-2.

(a) Determine and sketch $h[n]$, the impulse response of the system $H(z)$.

(b) A second system is shown in Figure P5.6-3, in which the signal $x[n]$ is first time compressed by a factor of 2 and then passed through an LTI system $G(z)$ to obtain $r[n]$.

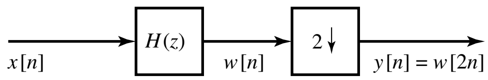


Figure P5.6-1

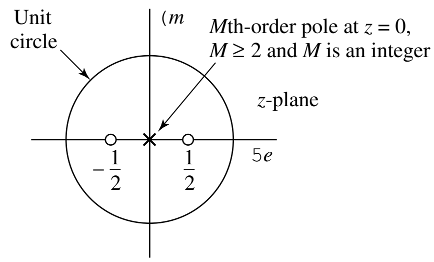


Figure P5.6-2

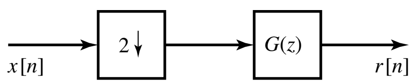


Figure P5.6-3

Determine whether $G(z)$ can be chosen so that $y[n] = r[n]$ for any input $x[n]$. If your answer is no, clearly explain. If your answer is yes, specify $G(z)$. If your answer depends on the value of M, clearly explain how. (M is constrained to be an integer greater than or equal to 2.)

5.7. Consider a linear time-invariant system whose system function is

$$H(z) = \frac{z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 3z^{-1}\right)}.$$

(a) Suppose the system is known to be stable. Determine the output $y[n]$ when the input $x[n]$ is the unit step sequence.

(b) Suppose the region of convergence of $H(z)$ includes $z = \infty$. Determine $y[n]$ evaluated at $n = 2$ when $x[n]$ is as shown in Figure P5.7-1.

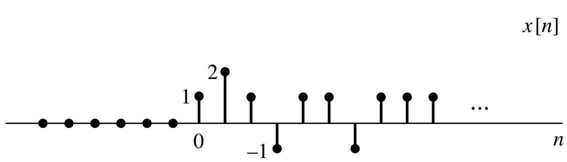


Figure P5.7-1

(c) Suppose we wish to recover $x[n]$ from $y[n]$ by processing $y[n]$ with an LTI system whose impulse response is given by $h_i[n]$. Determine $h_i[n]$. Does $h_i[n]$ depend on the region of convergence of $H(z)$?

5.8. A sequence $x[n]$ is the output of a linear time-invariant system whose input is $s[n]$. This system is described by the difference equation

$$x[n] = s[n] - e^{-8\alpha} s[n-8], \tag{P5.8-1}$$

where $0 < \alpha$.

(a) Find the system function

$$H_1(z) = \frac{X(z)}{S(z)},$$

and plot its poles and zeros in the z-plane. Indicate the region of convergence.

(b) We wish to recover $s[n]$ from $x[n]$ with a linear time-invariant system. Find the system function

$$H_2(z) = \frac{Y(z)}{X(z)}$$

such that $y[n] = s[n]$. Find all possible regions of convergence for $H_2(z)$, and for each, tell whether or not the system is causal and/or stable.

(c) Find all possible choices for the impulse response $h_2[n]$ such that

$$y[n] = h_2[n] * x[n] = s[n]. \tag{P5.8-2}$$

(d) For all choices determined in Part (c), demonstrate, by explicitly evaluating the convolution in Eq. (P5.8-2), that when $s[n] = \delta[n]$, $y[n] = \delta[n]$.

Note: As discussed in DTSP3E Problem 4.7, Eq. (P5.8-1) represents a simple model for a multipath channel. The systems determined in Parts (b) and (c), then, correspond to compensation systems to correct for the multipath distortion.

5.9. Consider a linear time-invariant system whose impulse response is

$$h[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{1}{3}\right)^n u[n].$$

The input $x[n]$ is zero for $n < 0$, but in general, may be nonzero for $0 \le n \le \infty$. We would like to compute the output $y[n]$ for $0 \le n \le 10^9$, and in particular, we want to compare the use of an FIR filter with that of an IIR filter for obtaining $y[n]$ over this interval.

(a) Determine the linear constant-coefficient difference equation for the IIR system relating $x[n]$ and $y[n]$.

(b) Determine the impulse response $h_1[n]$ of the minimum-length LTI FIR filter whose output $y_1[n]$ is identical to the output $y[n]$ for $0 \leq n \leq 10^9$.

(c) Specify the linear constant-coefficient difference equation associated with the FIR filter in Part (b).

(d) Compare the number of arithmetic operations (multiplications and additions) required to obtain $y[n]$ for $0 \leq n \leq 10^9$ using the linear constant-coefficient difference equations in Part (a) and in Part (c).

5.10. Consider a causal linear time-invariant system with system function $H(z)$ and real impulse response. $H(z)$ evaluated for $z = e^{j\omega}$ is shown in Figure P5.10-1.

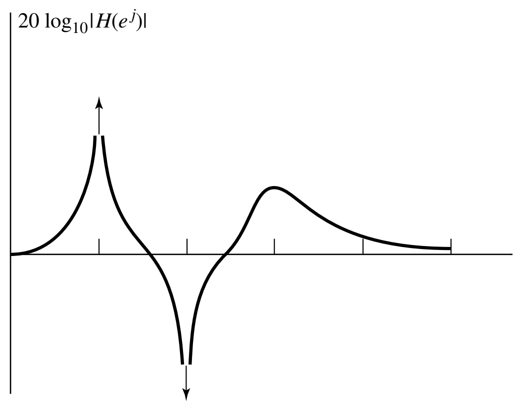


Figure P5.10-1

(a) Carefully sketch a pole–zero plot for $H(z)$ showing all information about the pole and zero locations that can be inferred from the figure.

(b) What can be said about the length of the impulse response?

(c) Specify whether $\angle H(e^{j\omega})$ is linear.

(d) Specify whether the system is stable.

5.11. Figure P5.11-1 shows two different interconnections of three systems. The impulse responses $h_1[n]$, $h_2[n]$, and $h_3[n]$ are as shown in Figure P5.11-2. Determine whether system A and/or system B is a generalized linear-phase system.

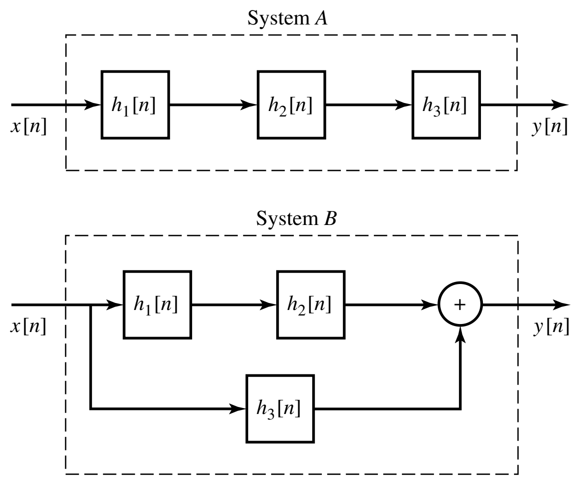


Figure P5.11-1

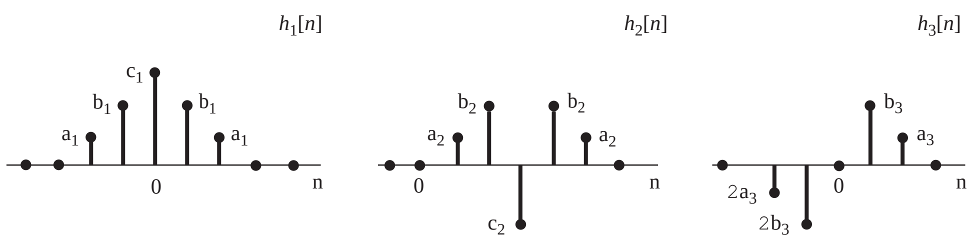


Figure P5.11-2

5.12. A causal linear time-invariant discrete-time system has system function

$$H(z) = \frac{(1 - 0.5z^{-1})(1 + 4z^{-2})}{(1 - 0.64z^{-2})}.$$

(a) Find expressions for a minimum-phase system $H_1(z)$ and an all-pass system $H_{\text{ap}}(z)$ such that

$$H(z) = H_1(z)H_{\text{ap}}(z).$$

(b) Find expressions for a different minimum-phase system $H_2(z)$ and a generalized linear-phase FIR system $H_{\text{lin}}(z)$ such that

$$H(z) = H_2(z)H_{\text{lin}}(z).$$

5.13. Consider an LTI system with input $x[n]$ and output $y[n]$. When the input to the system is

$$x[n] = 5\frac{\sin(0.4\pi n)}{\pi n} + 10\cos(0.5\pi n),$$

the corresponding output is

$$y[n] = 10\frac{\sin[0.3\pi(n-10)]}{\pi(n-10)}.$$

Determine the frequency response $H(e^{j\omega})$ and the impulse response $h[n]$ for the LTI system.

5.14. Figure P5.14-1 shows the pole–zero plots for three different causal LTI systems with real impulse responses. Indicate which of the following properties apply to each of the systems pictured: stable, IIR, FIR, minimum phase, all-pass, generalized linear phase, positive group delay at all ω.

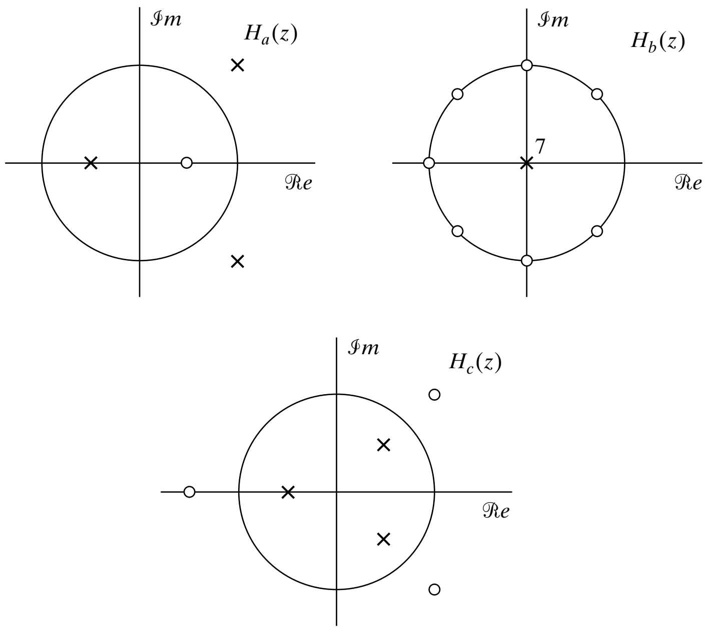


Figure P5.14-1

5.15. Let S_1 be an LTI system with system function:

$$H_1(z) = \frac{1-z^{-5}}{1-z^{-1}}, \qquad |z| > 0,$$

and impulse response $h_1[n]$.

(a) Is S_1 causal? Explain.

(b) Let $g[n] = h_1[n] * h_2[n]$. Specify an $h_2[n]$ such that $g[n]$ has at least nine nonzero samples and $g[n]$ can be considered the impulse response of a causal LTI system with strictly linear phase; i.e., $G(e^{j\omega}) = |G(e^{j\omega})|e^{-j\omega n_0}$ for some integer n_0.

(c) Let $q[n] = h_1[n] * h_3[n]$. Specify an $h_3[n]$ such that

$$q[n] = \delta[n] \qquad \text{for } 0 \le n \le 19.$$

5.16. The LTI systems $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ are generalized linear-phase systems. Which, if any, of the following systems also must be generalized linear-phase systems?

(a)

$$G_1(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

(b)

$$G_2(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})$$

(c)

$$G_3(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_1(e^{j\theta})H_2(e^{j(\omega-\theta)})d\theta$$

5.17. In this problem, you will consider three different LTI systems. All three are causal and have real impulse responses. You will be given additional information about each system. Using this information, state as much as possible about the poles and zeros of each system function and about the length of the impulse response of the system.

(a) $H_1(z)$ has a pole at $z = 0.9e^{j\pi/3}$, and when $x[n] = u[n]$, $\lim_{n\to\infty} y[n] = 0$.

(b) $H_2(z)$ has a zero at $z = 0.8e^{j\pi/4}$, $H_2(e^{j\omega})$ has linear phase with $\angle H_2(e^{j\omega}) = -2.5\omega$, and $20\log_{10}|H_2(e^{j0})| = -\infty$.

(c) $H_3(z)$ has a pole at $z = 0.8e^{j\pi/4}$ and $|H_3(e^{j\omega})| = 1$ for all ω.

5.18. The following three things are known about a signal $x[n]$ with z-transform $X(z)$:

(i) $x[n]$ is real valued and minimum phase,

(ii) $x[n]$ is zero outside the interval $0 \le n \le 4$,

(iii) $X(z)$ has a zero at $z = \frac{1}{2}e^{j\pi/4}$ and a zero at $z = \frac{1}{2}e^{j3\pi/4}$.

Based on this information, answer the following questions:

(a) Is $X(z)$ rational? Justify your answer.

(b) Sketch the complete pole–zero plot for $X(z)$ and specify its ROC.

(c) If $y[n] * x[n] = \delta[n]$ and $y[n]$ is rightsided, sketch the pole–zero plot for $Y(z)$ and specify its ROC.

5.19. Consider a real sequence $x[n]$ and its DTFT $X(e^{j\omega})$. Given the following information, determine and plot the sequence $x[n]$:

1. $x[n]$ is a finite-length sequence.

2. At $z = 0$, $X(z)$ has exactly five poles and no zeros. $X(z)$ *may* have poles or zeros at other locations.

3. The unwrapped phase function is

$$\arg\left[X(e^{j\omega})\right] = \begin{cases} -\alpha\omega + \frac{\pi}{2}, & 0 < \omega < \pi, \\ -\alpha\omega - \frac{\pi}{2}, & -\pi < \omega < 0, \end{cases}$$

for some real constant α.

4. The group delay of the sequence evaluated at $\omega = \frac{\pi}{2}$ is 2; i.e.,

$$\text{grd}\left[X(e^{j\omega})\right]_{\omega=\pi/2} = 2.$$

5.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 28.$$

6. If $y[n] = x[n] * u[n]$, then

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})d\omega &= 4, \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega}d\omega &= 6. \end{aligned}$$

7. $X(e^{j\omega})|_{\omega=\pi} = 0$.

8. The sequence $v[n]$ whose DTFT is $V(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$ satisfies $v[5] = -\frac{3}{2}$.

Chapter 6

Structures for discrete-time systems

Problems

6.1. Consider a causal linear time-invariant system whose system function is

$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{\left(1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2}\right)\left(1 + \frac{1}{4}z^{-1}\right)}.$$

(a) Draw the signal flow graphs for implementations of the system in each of the following forms:

(i) Direct form I

(ii) Direct form II

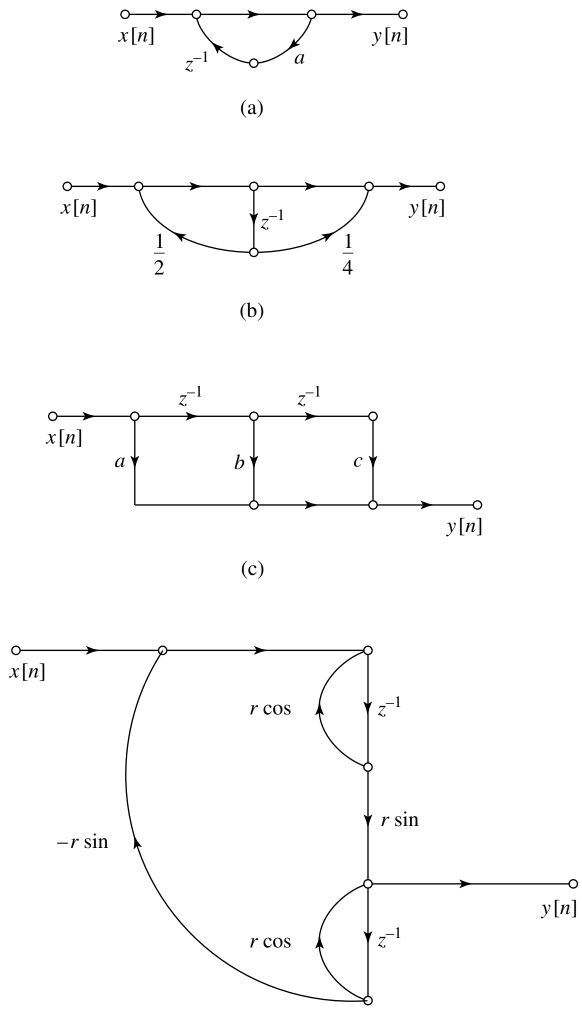
(iii) Cascade form using first- and second-order direct form II sections

(iv) Parallel form using first- and second-order direct form II sections

(v) Transposed direct form II

(b) Write the difference equations for the flow graph of (v) in Part (a), and show that this system has the correct system function.

6.2. Several flow graphs are shown in Figure P6.2-1. Determine the transpose of each flow graph, and verify that in each case the original and transposed flow graphs have the same system function.



(d)

Figure P6.2-1

6.3. Consider the system in Figure P6.3-1.

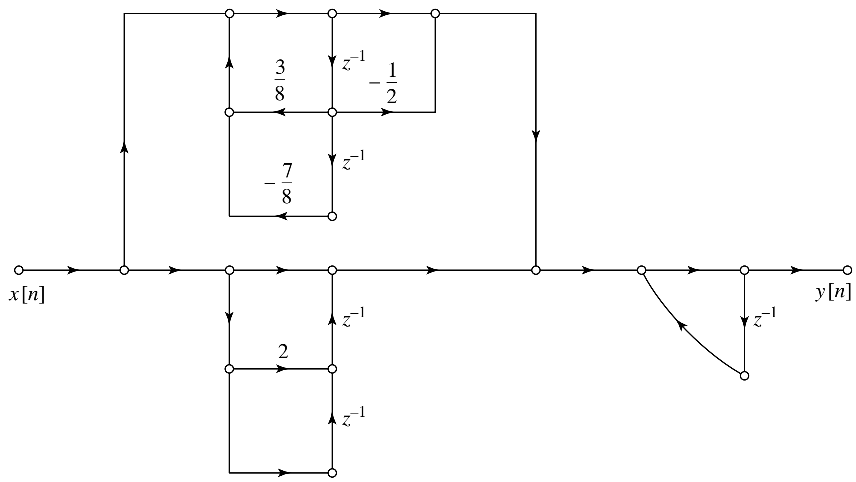


Figure P6.3-1

(a) Find the system function relating the z-transforms of the input and output.

(b) Write the difference equation that is satisfied by the input sequence $x[n]$ and the output sequence $y[n]$.

(c) Draw a signal flow graph that has the same input–output relationship as the system in Figure P6.3-1, but that has the smallest possible number of delay elements.

Chapter 7

Filter design techniques

Basic Problems

7.1. Suppose that we are given a continuous-time lowpass filter with frequency response $H_c(j\omega)$ such that

$$\begin{aligned} 1-\delta_1 \le |H_c(j\Omega)| &\le 1+\delta_1, \qquad |\Omega| \le \Omega_p, \\ |H_c(j\Omega)| &\le \delta_2, \qquad\quad |\Omega| \ge \Omega_s. \end{aligned}$$

A set of discrete-time lowpass filters can be obtained from $H_c(s)$ by using the bilinear transformation, i.e.,

$$H(z) = H_c(s)\big|_{s=(2/T_d)[(1-z^{-1})/(1+z^{-1})]},$$

with T_d variable.

(a) Assuming that Ω_p is fixed, find the value of T_d such that the corresponding passband cutoff frequency for the discrete-time system is $\omega_p = \pi/2$.

(b) With Ω_p fixed, sketch ω_p as a function of $0 < T_d < \infty$.

(c) With both Ω_p and Ω_s fixed, sketch the transition region $\Delta\omega = (\omega_s - \omega_p)$ as a function of $0 < T_d < \infty$.

7.2. A continuous-time filter with impulse response $h_c(t)$ and frequency-response magnitude

$$|H_c(j\Omega)| = \begin{cases} |\Omega|, & |\Omega| < 10\pi, \\ 0, & |\Omega| > 10\pi, \end{cases}$$

is to be used as the prototype for the design of a discrete-time filter. The resulting discrete-time system is to be used in the configuration of Figure P7.2-1 to filter the continuous-time signal $x_c(t)$.

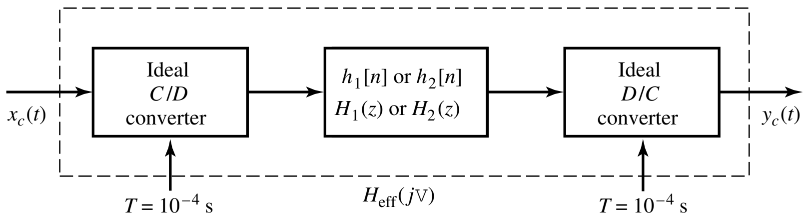


Figure P7.2-1

(a) A discrete-time system with impulse response $h_1[n]$ and system function $H_1(z)$ is obtained from the prototype continuous-time system by impulse invariance with $T_d = 0.01$; i.e., $h_1[n] = 0.01h_c(0.01n)$. Plot the magnitude of the overall effective frequency response $H_{\text{eff}}(j\Omega) = Y_c(j\Omega)/X_c(j\Omega)$ when this discrete-time system is used in Figure P7.2-1.

(b) Alternatively, suppose that a discrete-time system with impulse response $h_2[n]$ and system function $H_2(z)$ is obtained from the prototype continuous-time system by the bilinear transformation with $T_d = 2$; i.e.,

$$H_2(z) = H_c(s)\big|_{s=(1-z^{-1})/(1+z^{-1})}.$$

Plot the magnitude of the overall effective frequency response $H_{\text{eff}}(j\Omega)$ when this discrete-time system is used in Figure P7.2-1.

Advanced Problems

7.3. A discrete-time filter with system function $H(z)$ is designed by transforming a continuous-time filter with system function $H_c(s)$. It is desired that

$$H(e^{j\omega})\big|_{\omega=0} = H_c(j\Omega)\big|_{\Omega=0}.$$

(a) Could this condition hold for a filter designed by impulse invariance? If so, what condition(s), if any, would $H_c(j\Omega)$ have to satisfy?

(b) Could the condition hold for a filter designed using the bilinear transformation? If so, what conditions, if any, would $H_c(j\Omega)$ have to satisfy?

7.4. A discrete-time highpass filter can be obtained from a continuous-time lowpass filter by the following transformation:

$$H(z) = H_c(s)\big|_{s=[(1+z^{-1})/(1-z^{-1})]}.$$

(a) Show that this transformation maps the $j\Omega$-axis of the s-plane onto the unit circle of the z-plane.

(b) Show that if $H_c(s)$ is a rational function with all its poles inside the left-half s-plane, then $H(z)$ will be a rational function with all its poles inside the unit circle of the z-plane.

(c) Suppose a desired highpass discrete-time filter has specifications

$$\begin{aligned} |H(e^{j\omega})| &\leq 0.01, \quad |\omega| \leq \pi/3, \\ 0.95 \leq |H(e^{j\omega})| &\leq 1.05, \quad \pi/2 \leq |\omega| \leq \pi. \end{aligned}$$

Determine the specifications on the continuous-time lowpass filter so that the desired highpass discrete-time filter results from the transformation specified at the beginning of this problem.

7.5. An optimal equiripple FIR linear-phase filter was designed by the Parks–McClellan algorithm. The magnitude of its frequency response is shown in Figure P7.5-1. The maximum approximation error in the passband is $\delta_1 = 0.0531$, and the maximum approximation error in the stopband is $\delta_2 = 0.085$. The passband and stopband cutoff frequencies are $\omega_p = 0.4\pi$ and $\omega_s = 0.58\pi$.

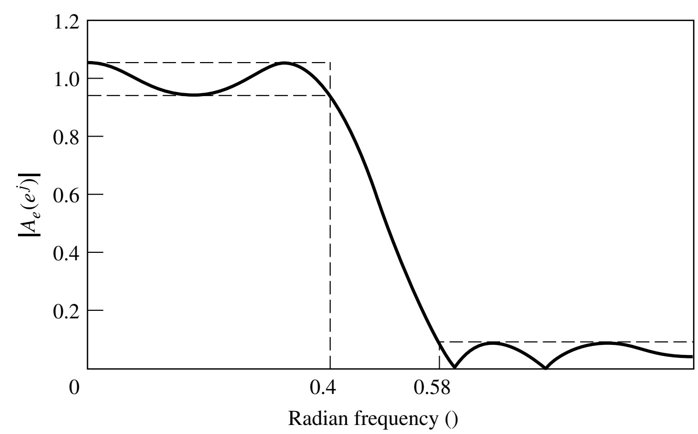


Figure P7.5-1

(a) What type (I, II, III, or IV) of linear-phase system is this? Explain how you can tell.

(b) What was the error-weighting function $W(\omega)$ that was used in the optimization?

(c) Carefully sketch the weighted approximation error; i.e., sketch

$$E(\omega) = W(\omega)[H_d(e^{j\omega}) - A_e(e^{j\omega})].$$

(Note that Figure P7.5-1 shows $|A_e(e^{j\omega})|$.)

(d) What is the length of the impulse response of the system?

(e) If this system is causal, what is the smallest group delay that it can have?

(f) Plot the zeros of the system function $H(z)$ as accurately as you can in the z-plane.

7.6. Filter C is a stable continuous-time IIR filter with system function $H_c(s)$ and impulse response $h_c(t)$. Filter B is a stable discrete-time filter with system function $H_b(z)$ and impulse response $h_b[n]$. Filter B is related to filter C through the bilinear transformation. Determine whether the following statement is true or false. If it is true, state your reasoning. If it is false, demonstrate with a counterexample.

Statement: Filter B cannot be an FIR filter.

7.7. Suppose that a discrete-time filter is obtained from a prototype continuous-time filter $H_e(s)$ by bilinear transformation. Furthermore, assume that the continuous-time filter has a constant group delay; i.e.,

$$H_e(j\Omega) = A(\Omega)e^{-j\Omega\alpha},$$

where $A(\Omega)$ is purely real. Would the resulting discrete-time filter also have a constant group delay? Explain your reasoning.

Chapter 8

The discrete Fourier transform

Problems

Basic Problems

8.1. $x[n]$ denotes a finite-length sequence of length N. Show that

$$x[((-n))_N] = x[((N-n))_N].$$

8.2. Consider a real finite-length sequence $x[n]$ with Fourier transform $X(e^{j\omega})$ and DFT $X[k]$. If

$$\mathcal{I}m\{X[k]\} = 0, \qquad k = 0, 1, \ldots, N-1,$$

can we conclude that

$$\mathcal{I}m\{X(e^{j\omega})\} = 0, \qquad -\pi \leq \omega \leq \pi?$$

State your reasoning if your answer is yes. Give a counterexample if your answer is no.

8.3. Consider the finite-length sequence $x[n]$ in Figure P8.3-1. The four-point DFT of $x[n]$ is denoted $X[k]$. Plot the sequence $y[n]$ whose DFT is

$$Y[k] = W_4^{3k} X[k].$$

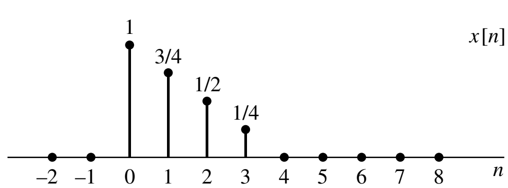


Figure P8.3-1

8.4. Consider the real finite-length sequence $x[n]$ shown in Figure P8.4-1.

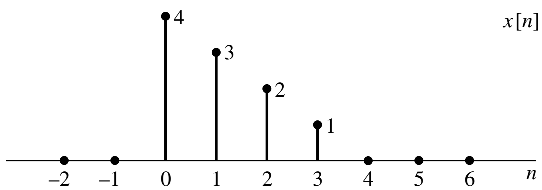


Figure P8.4-1

(a) Sketch the finite-length sequence $y[n]$ whose six-point DFT is

$$Y[k] = W_6^{4k} X[k],$$

where $X[k]$ is the six-point DFT of $x[n]$.

(b) Sketch the finite-length sequence $w[n]$ whose six-point DFT is

$$W[k] = \mathcal{R}e\{X[k]\}.$$

(c) Sketch the finite-length sequence $q[n]$ whose three-point DFT is

$$Q[k] = X[2k], \qquad k = 0, 1, 2.$$

8.5. Figure P8.5-1 shows two sequences,

$$x_1[n] = \begin{cases} 1, & 0 \le n \le 99, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$x_2[n] = \begin{cases} 1, & 0 \le n \le 9, \\ 0, & \text{otherwise.} \end{cases}$$

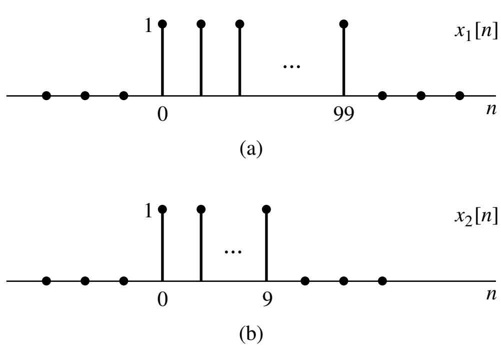


Figure P8.5-1

(a) Determine and sketch the linear convolution $x_1[n] * x_2[n]$.

(b) Determine and sketch the 100-point circular convolution $x_1[n] \circledast_{100} x_2[n]$.

(c) Determine and sketch the 110-point circular convolution $x_1[n] \circledast_{110} x_2[n]$.

Advanced Problems

8.6. Consider a finite-length sequence $x[n]$ of length N; i.e.,

$$x[n] = 0 \qquad \text{outside} \quad 0 \le n \le N-1.$$

$X(e^{j\omega})$ denotes the Fourier transform of $x[n]$. $\tilde{X}[k]$ denotes the sequence of 64 equally spaced samples of $X(e^{j\omega})$, i.e.,

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=2\pi k/64}.$$

It is known that in the range $0 \le k \le 63, \tilde{X}[32] = 1$ and all the other values of $\tilde{X}[k]$ are zero.

(a) If the sequence length is $N = 64$, determine one sequence $x[n]$ consistent with the given information. Indicate whether the answer is unique. If it is, clearly explain why. If it is not, give a second *distinct* choice.

(b) If the sequence length is $N = 192 = 3 \times 64$, determine one sequence $x[n]$ consistent with the constraint that in the range $0 \le k \le 63, \tilde{X}[32] = 1$ and all other values in that range are zero. Indicate whether the answer is unique. If it is, clearly explain why. If it is not, give a second *distinct* choice.

8.7. A finite-duration sequence $x[n]$ of length 8 has the eight-point DFT $X[k]$ shown in Figure P8.7-1. A new sequence of length 16 is defined by

$$y[n] = \begin{cases} x[n/2], & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

From the list in Figure P8.7-2, choose the sketch corresponding to $Y[k]$, the sixteen-point DFT of $y[n]$.

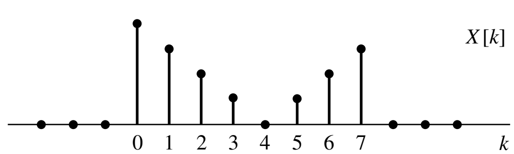


Figure P8.7-1

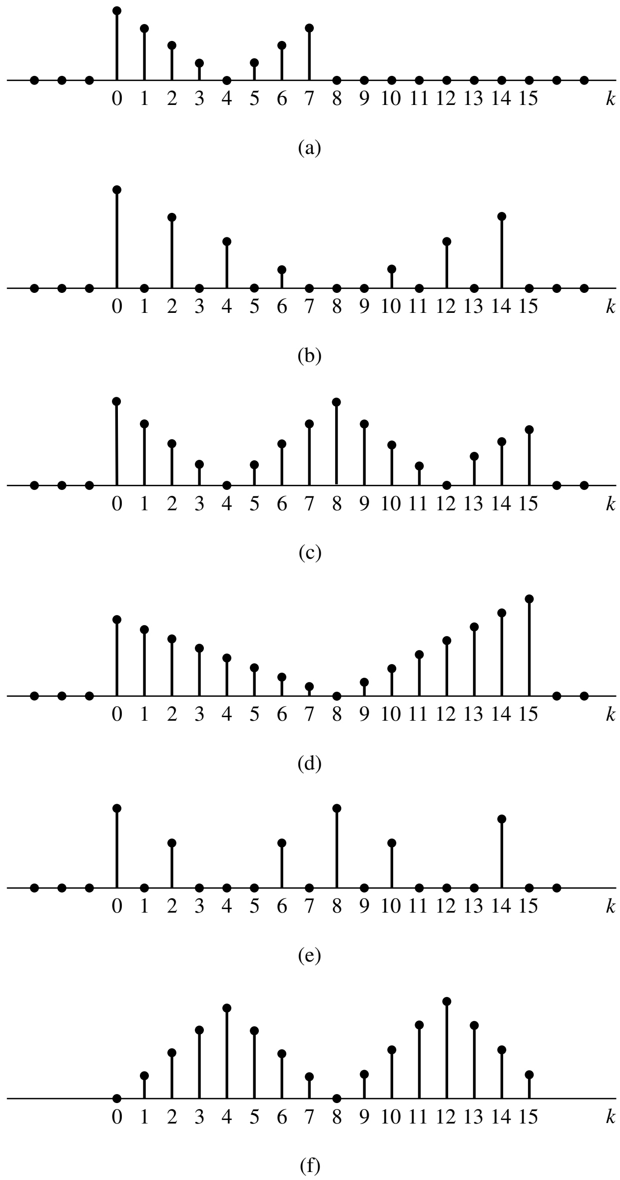


Figure P8.7-2

8.8. Consider a finite-length sequence $x[n]$ of length N as indicated in Figure P8.8-1. (The solid line is used to suggest the envelope of the sequence values between 0 and $N-1$.) Two finite-length sequences $x_1[n]$ and $x_2[n]$ of length $2N$ are constructed from $x[n]$ as indicated in Figure P8.8-2, with

$x_1[n]$ and $x_2[n]$ given as follows:

$$x_1[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise,} \end{cases}$$

$$x_2[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ -x[n-N], & N \le n \le 2N-1, \\ 0, & \text{otherwise.} \end{cases}$$

The N-point DFT of $x[n]$ is denoted by $X[k]$, and the $2N$-point DFTs of $x_1[n]$ and $x_2[n]$ are denoted by $X_1[k]$ and $X_2[k]$, respectively.

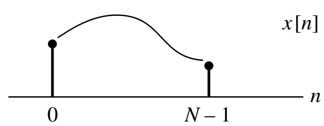


Figure P8.8-1

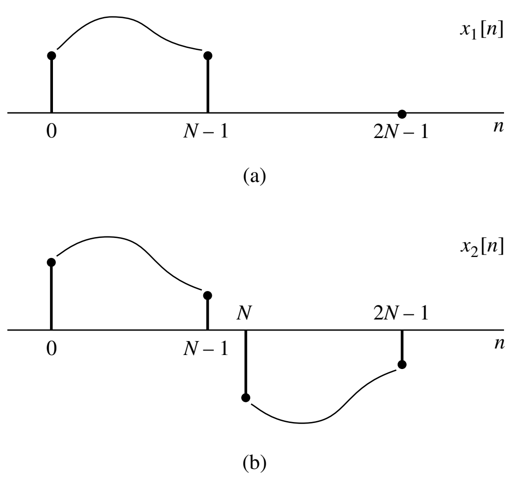


Figure P8.8-2

(a) Specify whether $X_2[k]$ can be obtained if $X[k]$ is given. Clearly indicate your reasoning.

(b) Determine the simplest possible relationship whereby one can obtain $X[k]$ from $X_1[k]$.

8.9. The even part of a real sequence $x[n]$ is defined by

$$x_e[n] = \frac{x[n] + x[-n]}{2}.$$

Suppose that $x[n]$ is a real finite-length sequence defined such that $x[n] = 0$ for $n < 0$ and $n \ge N$. Let $X[k]$ denote the N-point DFT of $x[n]$.

(a) Is $\mathcal{R}e\{X[k]\}$ the DFT of $x_e[n]$?

(b) What is the inverse DFT of $\mathcal{R}e\{X[k]\}$ in terms of $x[n]$?

8.10. Determine a sequence $x[n]$ that satisfies all of the following three conditions:
Condition 1: The Fourier transform of $x[n]$ has the form

$$X(e^{j\omega}) = 1 + A_1 \cos\omega + A_2 \cos 2\omega,$$

where A_1 and A_2 are some unknown constants.
Condition 2: The sequence $x[n]*\delta[n-3]$ evaluated at $n = 2$ is 5.
Condition 3: For the three-point sequence $w[n]$ shown in Figure P8.10-1, the result of the eight-point circular convolution of $w[n]$ and $x[n-3]$ is 11 when $n = 2$; i.e.,

$$\sum_{m=0}^{7} w[m]x[((n-3-m))_8]\bigg|_{n=2} = 11.$$

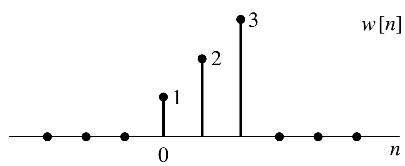


Figure P8.10-1

8.11. Consider the finite-length sequence

$$x[n] = 2\delta[n] + \delta[n-1] + \delta[n-3].$$

We perform the following operation on this sequence:

(i) We compute the five-point DFT $X[k]$.

(ii) We compute a five-point inverse DFT of $Y[k] = X[k]^2$ to obtain a sequence $y[n]$.

(a) Determine the sequence $y[n]$ for $n = 0, 1, 2, 3, 4$.

(b) If N-point DFTs are used in the two-step procedure, how should we choose N so that $y[n] = x[n]*x[n]$ for $0 \le n \le N-1$?

8.12. Consider a finite-duration sequence $x[n]$ that is zero for $n < 0$ and $n \ge N$, where N is even. The z-transform of $x[n]$ is denoted by $X(z)$. Table P8.37-1 lists seven sequences obtained from $x[n]$. Table P8.37-2 lists nine sequences obtained from $X(z)$. For each sequence in Table P8.37-1, find its DFT in Table P8.37-2. The size of the transform considered must be greater than or equal to the length of the sequence $g_k[n]$. *For purposes of illustration only,* assume that $x[n]$ can be represented by the envelope shown in Figure P8.12-1.

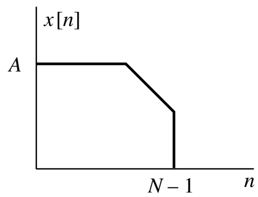
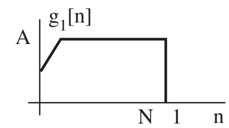
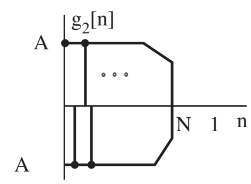
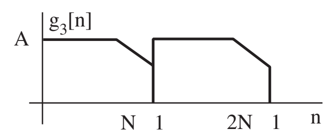


Figure P8.12-1

Table P8.37-1:

$g_1[n] = x[N-1-n]$	
$g_2[n] = (-1)^n x[n]$	
$g_3[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ x[n-N], & N \le n \le 2N-1, \\ 0, & \text{otherwise} \end{cases}$	

(*Continued*)

8.13. $x[n]$ is a real-valued finite-length sequence of length 10 and is nonzero in the interval from 0 to 9, i.e.,

$$\begin{aligned} x[n] &= 0, \qquad n < 0, n \ge 10, \\ x[n] &\ne 0, \qquad 0 \le n \le 9. \end{aligned}$$

$X(e^{j\omega})$ denotes the Fourier transform of $x[n]$, and $X[k]$ denotes the 10-point DFT of $x[n]$.
Determine a choice for $x[n]$ so that $X[k]$ is real valued for all k and

$$X(e^{j\omega}) = A(\omega)e^{j\alpha\omega}, \qquad |\omega| < \pi,$$

where $A(\omega)$ is real and α is a nonzero real constant.

8.14. Two finite-length sequences $x_1[n]$ and $x_2[n]$, which are zero outside the interval $0 \le n \le 99$, are circularly convolved to form a new sequence $y[n]$; i.e.,

$$y[n] = x_1[n] \textcircled{100}\, x_2[n] = \sum_{k=0}^{99} x_1[k]x_2[((n-k))_{100}], \qquad 0 \le n \le 99.$$

Table P8.37-1: (*Continued*)

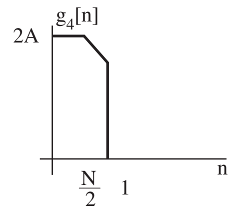
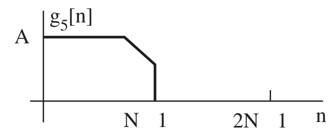
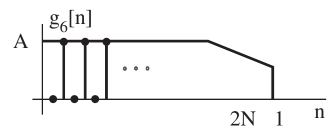
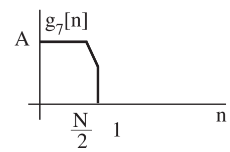
$g_4[n] = \begin{cases} x[n] + x[n+N/2], & 0 \le n \le N/2-1, \\ 0, & \text{otherwise} \end{cases}$	
$g_5[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ 0, & N \le n \le 2N-1, \\ 0, & \text{otherwise} \end{cases}$	
$g_6[n] = \begin{cases} x[n/2], & n \text{ even}, \\ 0, & n \text{ odd} \end{cases}$	
$g_7[n] = x[2n]$	

Table P8.37-2:

$H_1[k] = X(e^{j2\pi k/N})$
$H_2[k] = X(e^{j2\pi k/2N})$
$H_3[k] = \begin{cases} 2X(e^{j2\pi k/2N}), & k \text{ even}, \\ 0, & k \text{ odd} \end{cases}$
$H_4[k] = X(e^{j2\pi k/(2N-1)})$
$H_5[k] = 0.5\{X(e^{j2\pi k/N}) + X(e^{j2\pi(k+N/2)/N})\}$
$H_6[k] = X(e^{j4\pi k/N})$
$H_7[k] = e^{j2\pi k/N} X(e^{-j2\pi k/N})$
$H_8[k] = X(e^{j(2\pi/N)(k+N/2)})$
$H_9[k] = X(e^{-j2\pi k/N})$

If, in fact, $x_1[n]$ is nonzero only for $10 \le n \le 39$, determine the set of values of n for which $y[n]$ is guaranteed to be identical to the *linear* convolution of $x_1[n]$ and $x_2[n]$.

8.15. Consider two finite-length sequences $x[n]$ and $h[n]$ for which $x[n] = 0$ outside the interval $0 \le n \le 49$ and $h[n] = 0$ outside the interval $0 \le n \le 9$.

(a) What is the maximum possible number of nonzero values in the *linear* convolution of $x[n]$ and $h[n]$?

(b) The 50-point *circular* convolution of $x[n]$ and $h[n]$ is

$$x[n] \circledast_{50} h[n] = 10, \qquad 0 \le n \le 49.$$

The first 5 points of the *linear* convolution of $x[n]$ and $h[n]$ are

$$x[n] * h[n] = 5, \qquad 0 \le n \le 4.$$

Determine as many points as possible of the linear convolution of $x[n] * h[n]$.

8.16. Consider two finite-duration sequences $x[n]$ and $y[n]$. $x[n]$ is zero for $n < 0$, $n \ge 40$, and $9 < n < 30$, and $y[n]$ is zero for $n < 10$ and $n > 19$, as indicated in Figure P8.16-1.

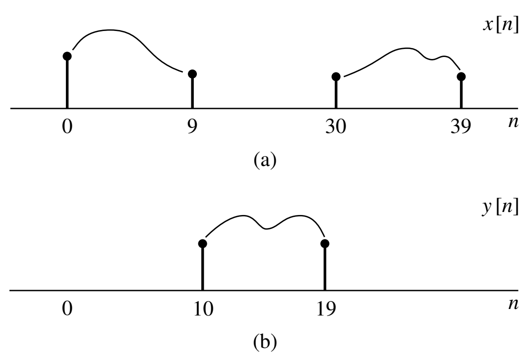


Figure P8.16-1

Let $w[n]$ denote the linear convolution of $x[n]$ and $y[n]$. Let $g[n]$ denote the 40-point circular convolution of $x[n]$ and $y[n]$:

$$w[n] = x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k],$$

$$g[n] = x[n] \circledast_{40} y[n] = \sum_{k=0}^{39} x[k]y[((n-k))_{40}].$$

(a) Determine the values of n for which $w[n]$ can be nonzero.

(b) Determine the values of n for which $w[n]$ can be obtained from $g[n]$. Explicitly specify at what index values n in $g[n]$ these values of $w[n]$ appear.

8.17. Two finite-duration sequences $h_1[n]$ and $h_2[n]$ of length 8 are sketched in Figure P8.17-1. The two sequences are related by a circular shift, i.e., $h_1[n] = h_2[((n - m))_8]$.

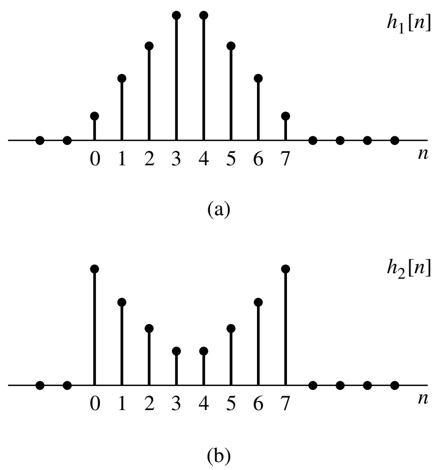


Figure P8.17-1

(a) Specify whether the magnitudes of the eight-point DFTs are equal.

(b) We wish to implement a lowpass FIR filter and must use either $h_1[n]$ or $h_2[n]$ as the impulse response. Which one of the following statements is correct?

(i) $h_1[n]$ is a better lowpass filter than $h_2[n]$.

(ii) $h_2[n]$ is a better lowpass filter than $h_1[n]$.

(iii) The two sequences are both about equally good (or bad) as lowpass filters.

8.18. We want to implement the linear convolution of a 10,000-point sequence with a finite impulse response that is 100 points long. The convolution is to be implemented by using DFTs and inverse DFTs of length 256.

(a) If the overlap–add method is used, what is the minimum number of 256-point DFTs and the minimum number of 256-point inverse DFTs needed to implement the convolution for the entire 10,000-point sequence? Justify your answer.

(b) If the overlap–save method is used, what is the minimum number of 256-point DFTs and the minimum number of 256-point inverse DFTs needed to implement the convolution for the entire 10,000-point sequence? Justify your answer.

(c) We will see in DTSP3E Chapter 9 that when N is a power of 2, an N-point DFT or inverse DFT requires $(N/2)\log_2 N$ complex multiplications and $N\log_2 N$ complex additions. For the same filter and impulse response length considered in Parts (a) and (b), compare the number of arithmetic operations (multiplications and additions) required in the overlap–add method, the overlap–save method, and direct convolution.

8.19. Let $x_1[n]$ be a sequence obtained by expanding the sequence $x[n] = \left(\frac{1}{4}\right)^n u[n]$ by a factor of 4; i.e,

$$x_1[n] = \begin{cases} x[n/4], & k = 0, \pm 4, \pm 8, \ldots, \\ 0, \text{otherwise.} \end{cases}$$

Find and sketch a six-point sequence $q[n]$ whose six-point DFT $Q[k]$ satisfies the two constraints

$$\begin{aligned} Q[0] &= X_1(1), \\ Q[3] &= X_1(-1), \end{aligned}$$

where $X_1(z)$ represents the z-transform of $x_1[n]$.

8.20. Let $x_2[n]$ be a real-valued five-point sequence whose seven-point DFT is denoted by $X_2[k]$. If Real$\{X_2[k]\}$ is the seven-point DFT of $g[n]$, show that $g[0] = x_2[0]$, and determine the relationship between $g[1]$ and $x_2[1]$. Justify your answer.

8.21. Shown in Figure P8.21-1 are three finite-length sequences of length 5. $X_i(e^{j\omega})$ denotes the DTFT of $x_i[n]$, and $X_i[k]$ denotes the five-point DFT of $x_i[n]$. For each of the following properties, indicate which sequences satisfy the property and which do not. Clearly justify your answers for each sequence and each property.

(i) $X_i[k]$ is real for all k.

(ii) $X_i(e^{j\omega}) = A_i(\omega)e^{j\alpha_i\omega}$, where $A_i(\omega)$ is real and α_i is constant.

(iii) $X_i[k] = B_i[k]e^{j\gamma_i k}$ where $B_i[k]$ is real and γ_i is a constant.

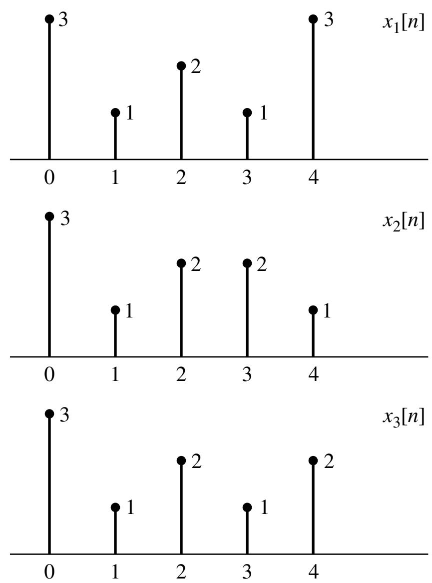


Figure P8.21-1

8.22. (a) Suppose

$$x[n] = 0, \qquad n < 0, \quad n > (N-1),$$

is an N-point sequence having at least one nonzero sample. Is it possible for such a sequence to have a DTFT

$$X(e^{j2\pi k/M}) = 0, \qquad k = 0, 1, \ldots, M-1,$$

where M is an integer greater than or equal to N? If your answer is yes, construct an example. If your answer is no, explain your reasoning.

(b) Suppose $M < N$. Repeat Part (a).

8.23. Suppose $x_1[n]$ is an infinite-length, stable (i.e., absolutely summable) sequence with z-transform given by

$$X_1(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}.$$

Suppose $x_2[n]$ is a finite-length sequence of length N, and the N-point DFT of $x_2[n]$ is

$$X_2[k] = X_1(z)\big|_{z=e^{j2\pi k/N}}, \qquad k = 0, 1, \ldots, N-1.$$

Determine $x_2[n]$.

Chapter 9

Computation of the discrete Fourier transform

Problems

Advanced Problems

9.1. The N-point DFT of the N-point sequence $x[n] = e^{-j(\pi/N)n^2}$, for N even, is

$$X[k] = \sqrt{N} e^{-j\pi/4} e^{j(\pi/N)k^2}.$$

Determine the $2N$-point DFT of the $2N$-point sequence $y[n] = e^{-j(\pi/N)n^2}$, assuming that N is even.

9.2. Suppose that a finite-length sequence $x[n]$ has the N-point DFT $X[k]$, and suppose that the sequence satisfies the symmetry condition

$$x[n] = -x[((n + N/2))_N], \qquad 0 \leq n \leq N-1,$$

where N is even and $x[n]$ is complex.

(a) Show that $X[k] = 0$ for $k = 0, 2, \ldots, N-2$.

(b) Show how to compute the odd-indexed DFT values $X[k], k = 1, 3, \ldots, N-1$ using only one $N/2$-point DFT plus a small amount of extra computation.

Chapter 10

Fourier analysis of signals using the discrete Fourier transform

Problems

Advanced Problems

10.1. Suppose that $y[n]$ is the output of a linear time-invariant FIR system with input $x[n]$; i.e.,

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k].$$

(a) Obtain a relationship between the time-dependent Fourier transform $Y[n, \lambda)$ of the output of the linear system and the time-dependent Fourier transform $X[n, \lambda)$ of the input.

(b) Show that if the window is long compared to M, then

$$\check{Y}[n, \lambda) \simeq H(e^{j\lambda})\check{X}[n, \lambda),$$

where $H(e^{j\omega})$ is the frequency response of the linear system.